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Balanced 0, ± Matrices
Part I: Decomposition

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Abstract

A $0, \pm 1$ matrix is *balanced* if, in every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. This paper extends the decomposition of balanced $0, 1$ matrices obtained by Conforti, Cornuéjols and Rao to the class of balanced $0, \pm 1$ matrices. As a consequence, we obtain a polynomial time algorithm for recognizing balanced $0, \pm 1$ matrices.

1 Introduction

A $0,1$ matrix is *balanced* if it does not contain a square submatrix of odd order with two ones per row and column. This notion was introduced by Berge [1] and extended to $0,\pm1$ matrices by Truemper [19].

A $0,\pm1$ matrix is *balanced* if, in every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. This paper extends the decomposition of balanced $0,1$ matrices obtained by Conforti, Cornuéjols and Rao [7] to the class of balanced $0,\pm1$ matrices. As a consequence, we obtain a polynomial time algorithm for recognizing balanced $0,\pm1$ matrices. This algorithm extends the algorithm in [7] for recognizing balanced $0,1$ matrices. It is discussed in a sequel paper.

The class of balanced $0,\pm1$ matrices properly includes totally unimodular $0,\pm1$ matrices. (A matrix is *totally unimodular* if every square submatrix has determinant equal to $0,\pm1$.) The fact that every totally unimodular matrix is balanced is implied, for example, by Camion's theorem [3] which states that a $0,\pm1$ matrix is totally unimodular if and only if, in every square submatrix with an *even number* of nonzero entries per row and column, the sum of the entries is a multiple of four. Therefore our work can also be viewed as an extension of Seymour's decomposition and recognition of totally unimodular matrices [18].

In Section 3 we show that, to understand the structure of balanced $0,\pm1$ matrices, it is sufficient to understand the structure of the zero-nonzero pattern i.e. the $0,1$ matrices that can be signed to be balanced. Such $0,1$ matrices are said to be *balanceable*. Clearly balanced $0,1$ matrices are balanceable but the converse is not true: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is balanceable but not balanced. Section 4 describes the cutsets used in our decomposition theorem and Section 5 states the theorem and outlines its proof. In Section 6, we relate our result to Seymour's [18] decomposition theorem for totally unimodular matrices. The proofs are given in Sections 8 - 13. The necessary definitions and notation are introduced in Section 7.

Interestingly, a number of polyhedral results known for balanced $0,1$ matrices and totally unimodular matrices can be generalized to balanced $0,\pm1$ matrices. It follows that several problems in propositional logic can be solved in polynomial time by linear programming when the underlying clauses are

"balanced". These results are reviewed in Section 2.

2 Bicoloring, Polyhedra and Propositional Logic

Berge [1] introduced the following notion. A $0,1$ matrix is *bicolorable* if its columns can be partitioned into blue and red columns in such a way that every row with two or more $1's$ contains a 1 in a blue column and a 1 in a red column. This notion provides the following characterization of balanced $0,1$ matrices.

Theorem 2.1 (Berge [1]) *A $0,1$ matrix A is balanced if and only if every submatrix of A is bicolorable.*

Ghouila-Houri [16] introduced the notion of *equitable bicoloring* for a $0,\pm 1$ matrix A as follows. The columns of A are partitioned into blue columns and red columns in such a way that, for every row of A, the sum of the entries in the blue columns differs from the sum of the entries in the red columns by at most one.

Theorem 2.2 (Ghouila-Houri [16]) *A $0,\pm 1$ matrix A is totally unimodular if and only if every submatrix of A has an equitable bicoloring.*

A $0,\pm 1$ matrix A is *bicolorable* if its columns can be partitioned into blue columns and red columns in such a way that every row with two or more nonzero entries either contains two entries of opposite sign in columns of the same color, or contains two entries of the same sign in columns of different colors. For a $0,1$ matrix, this definition coincides with Berge's notion of bicoloring. Clearly, if a $0,\pm 1$ matrix has an equitable bicoloring as defined by Ghouila-Houri, then it is bicolorable.

Theorem 2.3 (Conforti, Cornuéjols [6]) *A $0,\pm 1$ matrix A is balanced if and only if every submatrix of A is bicolorable.*

Balanced $0,1$ matrices are important in integer programming due to the fact that several polytopes, such as the set covering, packing and partitioning polytopes, only have integral extreme points when the constraint matrix is

balanced. Such integrality results were first observed by Berge [2] and then expanded upon by Fulkerson, Hoffman and Oppenheim [14]. In the case of balanced $0, \pm 1$ matrices, similar integrality results were proved by Conforti and Cornuéjols [6] for the generalized set covering, packing and partitioning polytopes.

Given a $0, \pm 1$ matrix A, let $n(A)$ denote the column vector whose i^{th} component is the number of -1's in the i^{th} row of matrix A.

Theorem 2.4 (Conforti, Cornuéjols [6]) *Let M be a $0, \pm 1$ matrix. Then the following statements are equivalent:*
(i) M is balanced.
(ii) For each submatrix A of M, the generalized set covering polytope $\{x : Ax \geq \mathbf{1} - n(A),\ \mathbf{0} \leq x \leq \mathbf{1}\}$ is integral.
(iii) For each submatrix A of M, the generalized set packing polytope $\{x : Ax \leq \mathbf{1} - n(A),\ \mathbf{0} \leq x \leq \mathbf{1}\}$ is integral.
(iv) For each submatrix A of M, the generalized set partitioning polytope $\{x : Ax = \mathbf{1} - n(A),\ \mathbf{0} \leq x \leq \mathbf{1}\}$ is integral.

Several problems in propositional logic can be written as generalized set covering problems. For example, the satisfiability problem in conjunctive normal form (SAT) is to find whether the formula

$$\bigwedge_{i \in S} \Big(\bigvee_{j \in P_i} x_j \vee \bigvee_{j \in N_i} \neg x_j \Big)$$

is true. This is the case if and only if the system of inequalities

$$\sum_{j \in P_i} x_j - \sum_{j \in N_i} (1 - x_j) \geq 1 \text{ for all } i \in S$$

has a $0, 1$ solution vector x. This is a generalized set covering problem

$$\begin{aligned} &Ax \geq \mathbf{1} - n(A) \\ &x \in \{0, 1\}^n. \end{aligned}$$

Given a set of clauses $\bigvee_{j \in P_i} x_j \vee \bigvee_{j \in N_i} \neg x_j$ with weights w_i, MAXSAT consists of finding a truth assignment which satisfies a maximum weight set of clauses. MAXSAT can be formulated as the integer program

$$\begin{array}{ll} Min & \sum_{i=1}^{m} w_i s_i \\ & Ax + s \geq 1 - n(A) \\ & x \in \{0,1\}^n, s \in \{0,1\}^m. \end{array}$$

Similarly, the inference problem in propositional logic can be formulated as

$$min\ \{cx : Ax \geq \mathbf{1} - n(A),\ x \in \{0,1\}^n\}.$$

The above three problems are NP-hard in general but SAT and logical inference can be solved efficiently for Horn clauses, clauses with at most two literals and several related classes [4],[20]. MAXSAT remains NP-hard for Horn clauses with at most two literals [15]. A consequence of Theorem 2.4 is the following.

Corollary 2.5 *SAT, MAXSAT and logical inference can be solved in polynomial time by linear programming when the corresponding* $0, \pm 1$ *matrix* A *is balanced.*

In fact SAT and logical inference can be solved by repeated application of unit resolution when the underlying $0, \pm 1$ matrix A is balanced [5].

3 Balanceable $0, 1$ Matrices

In this section, we consider the following question: given a $0,1$ matrix, is it possible to turn some of the 1's into -1's in order to obtain a balanced $0, \pm 1$ matrix? A $0,1$ matrix for which such a signing exists is called a *balanceable* matrix. It turns out that in order to understand the structure of balanced $0, \pm 1$ matrices, it is sufficient to concentrate on the zero-nonzero pattern, i.e. it is sufficient to understand the structure of the $0,1$ matrices that are balanceable. In fact, if a $0,1$ matrix is balanceable, there is a simple algorithm (which we state later) to perform the signing into a balanced $0, \pm 1$ matrix. So, in effect, the problem of recognizing whether a $0,1$ matrix is balanceable is equivalent to the problem of recognizing whether a given $0, \pm 1$ matrix is balanced.

Given a $0,1$ matrix A, the *bipartite graph representation of* A is the bipartite graph G having a node in V^r for every row of A, a node in V^c for every column of A and an edge ij joining nodes $i \in V^r$ and $j \in V^c$ if and

only if the entry a_{ij} of A equals 1. The sets V^r and V^c are the *sides* of the bipartition. We say that G is balanced if A is.

A ***signed graph*** G is a graph together with an assignment of weights $+1, -1$ to the edges of G. To a $0, \pm 1$ matrix corresponds its signed bipartite graph representation. A signed bipartite graph G is *balanced* if it is the signed bipartite graph representation of a balanced $0, \pm 1$ matrix. Thus a signed bipartite graph G is balanced if and only if, in every hole H of G, the sum of the weights of the edges in H is a multiple of four. (A *hole* in a graph is a chordless cycle).

A bipartite graph G is *balanceable* if there exists a signing of its edges so that the resulting signed graph is balanced.

Remark 3.1 *Since cuts and cycles of a graph G have even intersection, it follows that, if a signed bipartite graph G is balanced, then the signed bipartite graph G', obtained by switching signs on the edges of a cut, is also balanced.*

For every edge uv of a spanning tree, there is a cut containing uv and no other edge of the tree (such cuts are known as *fundamental cuts*), and every cut is a symmetric difference of fundamental cuts. Thus, if G is a balanceable bipartite graph, its signing into a balanced bipartite graph is unique up to the (arbitrary) signing of a spanning tree of G. This was already observed by Camion [3] in the context of $0, 1$ matrices that can be signed to be totally unimodular. So Remark 3.1 implies that a bipartite graph G is balanceable if and only if the following signing algorithm produces a balanced signed bipartite graph:

Signing Algorithm

Choose a spanning tree of G, sign its edges arbitrarily and recursively choose an edge uv which closes a hole H of G with the previously chosen edges, and sign uv so that the sum of the weights of the edges in H is a multiple of four.

Note that, in the signing algorithm, the edge uv can be chosen to close the smallest length hole with the previously chosen edges. Such a hole H is also a hole in G, else a chord of H in G contradicts the choice of uv.

It follows from this signing algorithm, and the uniqueness of the signing (up to the signing of a spanning tree), that the problem of recognizing

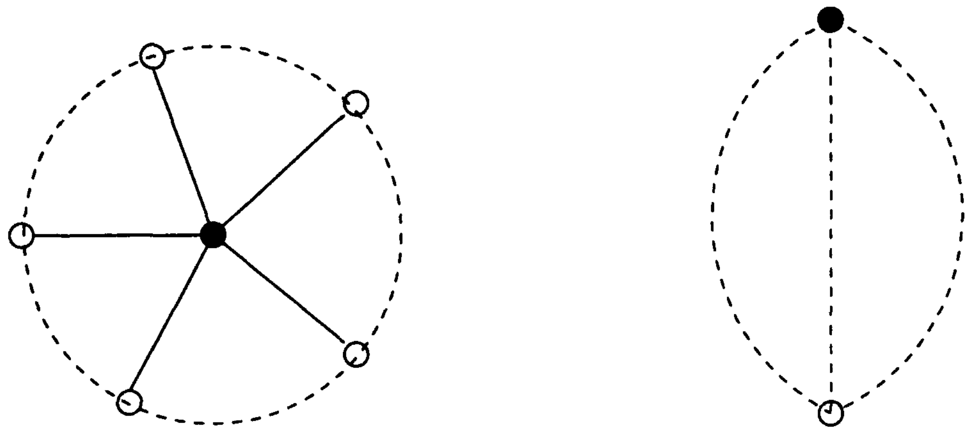


Figure 1: Odd wheel and 3-path configuration

whether a bipartite graph is balanceable is equivalent to the problem of recognizing whether a signed bipartite graph is balanced.

Let G be a bipartite graph. Let u, v be two nonadjacent nodes in opposite sides of the bipartition. A *3-path configuration connecting u and v*, denoted by $3PC(u,v)$, is defined by three chordless paths P_1, P_2, P_3 connecting u and v, having no common intermediate nodes and such that the subgraph induced by the nodes of these three paths contains no other edges than those of the paths (see Figure 1). Since paths P_1, P_2, P_3 of a 3-path configuration are of length one or three modulo four, the sum of the weights of the edges in each path is also one or three modulo four. It follows that two of the three paths induce a hole of weight two modulo four. So a bipartite graph which contains a 3-path configuration as an induced subgraph is not balanceable.

A *wheel*, denoted by (H, x), is defined by a hole H and a node $x \notin V(H)$ having at least three neighbors in H, say $x_1, x_2, \ldots, x_n$. If n is even, the wheel is an *even wheel*, otherwise it is an *odd wheel* (see Figure 1). An edge xx_i is a *spoke*. A subpath of H connecting x_i and x_j is called a *sector* if it contains no intermediate node x_l, $1 \leq l \leq n$. Consider a wheel which is signed to be balanced. By Remark 3.1, all spokes of the wheel can be assumed to be signed positive. This implies that the sum of the weights of the edges in each sector is two modulo four. Hence the wheel must be an even wheel.

So, balanceable bipartite graphs contain neither odd wheels nor 3-path

configurations. This fact is extensively used in our proofs in this paper. The following important theorem of Truemper [19] states that the converse is also true.

Theorem 3.2 (Truemper [19]) *A bipartite graph is balanceable if and only if it does not contain an odd wheel or a 3-path configuration.*

4 Cutsets

In this section we introduce the operations needed for our decomposition result. A set S of nodes (edges) of a connected graph G is a *node (edge) cutset* if the subgraph $G \setminus S$, obtained from G by removing the nodes (edges) in S, is disconnected.

Extended Star Cutsets

A *biclique* is a complete bipartite graph containing at least one node from each side of the bipartition and it is denoted by K_{BD} where B and D are the sets of nodes in the two sides of the bipartition.

For a node x, let $N(x)$ denote the set of all neighbors of x. In a bipartite graph G, an *extended star* $(x; T; A; N)$ is defined by disjoint subsets T, A, N of $V(G)$ and a node $x \in T$ such that

(i) $A \cup N \subseteq N(x)$,

(ii) the node set $T \cup A$ induces a biclique (with node set T on one side of the bipartition and node set A on the other),

(iii) if $|T| \geq 2$, then $|A| \geq 2$.

This concept was introduced in [7]. An *extended star cutset* is one where $T \cup A \cup N$ is a node cutset.

Joins

Let K_{BD} be a biclique with the property that its edge set $E(K_{BD})$ is a cutset of the connected bipartite graph G and no connected component of $G \setminus E(K_{BD})$ contains both a node of B and a node of D. Let G_B be the union of the components of $G \setminus E(K_{BD})$ containing a node of B. Similarily, let G_D be the union of the components of $G \setminus E(K_{BD})$ containing a node of

D. The set $E(K_{BD})$ is a *1-join* if the graphs G_B and G_D each contains at least two nodes. This concept was introduced by Cunningham and Edmonds [12].

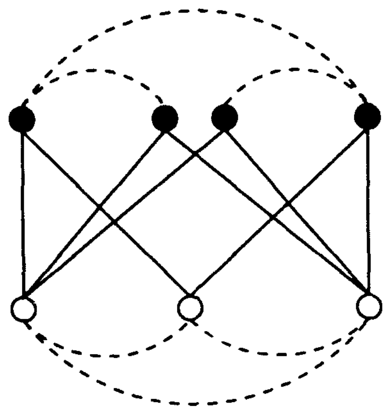
Let K_{BD} and K_{EF} be two bicliques of a connected bipartite graph G, where B, D, E, F are disjoint node sets and neither $E(K_{BD})$ nor $E(K_{EF})$ is a 1-join in G. Further assume that no connected component of $G \setminus E(K_{BD}) \cup E(K_{EF})$ has a node in B and one in D, or a node in E and one in F. Then, we can assume that every component of $G \setminus E(K_{BD}) \cup E(K_{EF})$ contains either a node of B and a node of E or a node of D and a node of F. Let G_{BE} be the union of the components of $G \setminus E(K_{BD}) \cup E(K_{EF})$ containing a node of B and a node of E. Similarily, let G_{DF} be the union of the components in $G \setminus E(K_{BD}) \cup E(K_{EF})$ containing a node of D and a node of F. The set $E(K_{BD}) \cup E(K_{EF})$ is a *2-join* if neither of the graphs G_{BE} and G_{DF} is a chordless path with all its intermediate nodes in $V(G) \setminus B \cup D \cup E \cup F$. This concept was introduced by Cornuéjols and Cunningham [11].

In a connected bipartite graph G, let A_i, $i = 1, \ldots, 6$ be disjoint nonempty node sets such that, for each i, every node in A_i is adjacent to every node in $A_{i-1} \cup A_{i+1}$ (indices are taken modulo 6), and these are the only edges in the subgraph A induced by the node set $\cup_{i=1}^{6} A_i$. Assume that $E(A)$ is an edge cutset but that no subset of its edges forms a 1-join or a 2-join. Furthermore assume that no connected component of $G \setminus E(A)$ contains a node in $A_1 \cup A_3 \cup A_5$ and a node in $A_2 \cup A_4 \cup A_6$. Let $G_{1,3,5}$ be the union of the components of $G \setminus E(A)$ containing a node in $A_1 \cup A_3 \cup A_5$ and $G_{2,4,6}$ be the union of components containing a node in $A_2 \cup A_4 \cup A_6$. The set $E(A)$ constitutes a *6-join* if the graphs $G_{1,3,5}$ and $G_{2,4,6}$ each contains at least four nodes (see Figure 2). This concept is new.

5 The Main Theorem

A bipartite graph is *restricted balanceable* if its edges can be signed so that the sum of the weights in each cycle is a multiple of four. Restricted balanceable bipartite graphs can be recognized in polynomial time [9], [22]. R_{10} is the balanceable bipartite graph defined by the cycle $x_1, \ldots, x_{10}, x_1$ of length 10 with chords $x_i x_{i+5}$, $1 \leq i \leq 5$ (see Figure 3).

We can now state the decomposition theorem for balanceable bipartite



Figur: 2: A 6-join

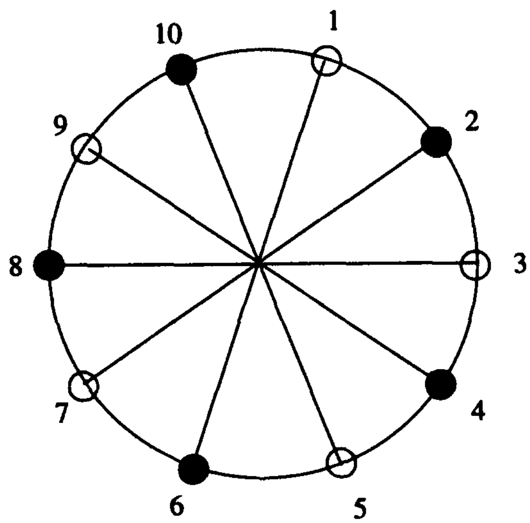


Figure 3: R_{10}

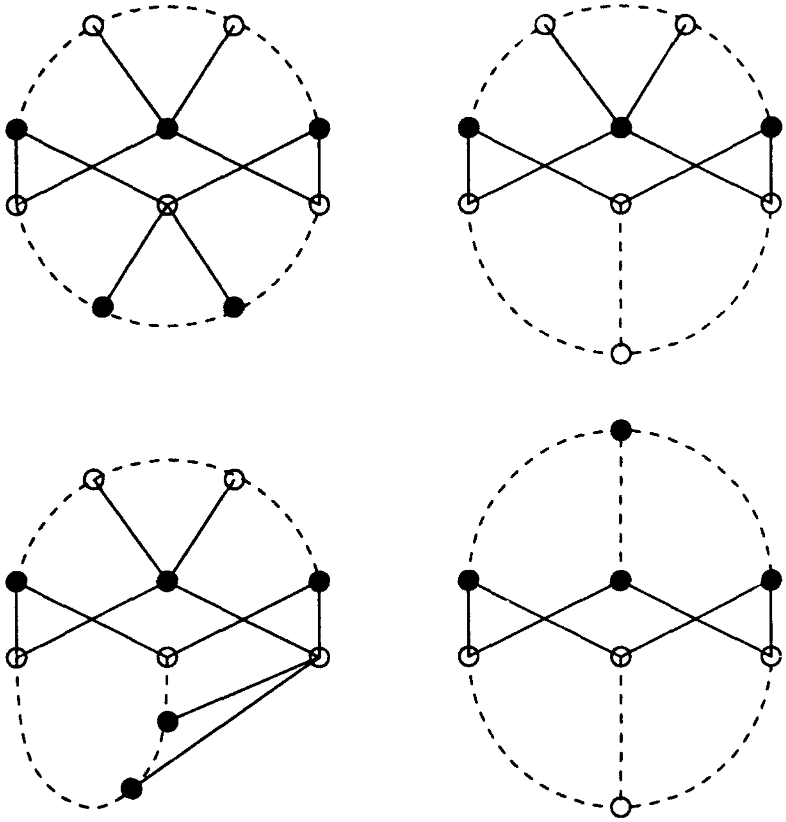


Figure 4: Four kinds of Connected 6-holes

graphs:

Theorem 5.1 *A balanceable bipartite graph that is not restricted balanceable is either R_{10} or contains a 2-join, a 6-join or an extended star cutset.*

The key idea in the proof of Theorem 5.1 is that if a balanceable bipartite graph G is not restricted balanceable, then one of the three following cases occurs: (i) the graph G contains R_{10} or (ii) it contains a certain induced subgraph which forces a 6-join or an extended star cutset of G, or (iii) an earlier result of Conforti, Cornuéjols and Rao [7] applies.

Connected 6-Holes

A *triad* consists of three internally node-disjoint paths $t, \dots, u$; $t, \dots, v$ and $t, \dots, w$, where t, u, v, w are distinct nodes and u, v, w belong to the same side of the bipartition. Furthermore, the graph induced by the nodes of the triad contains no other edges than those of the three paths. Nodes u, v and w are called the *attachments* and t is called the *meet* of the triad.

A *fan* consists of a chordless path $x, \dots, y$ together with a node z adjacent to at least one node of the path, where x, y and z are distinct nodes all belonging to the same side of the bipartition. Nodes x, y and z are called the *attachments* of the fan and z is the *center*. A *spoke* is an edge connecting z to a node of the fan.

A *connected 6-hole* Σ is a bipartite graph induced by two disjoint node sets $T(\Sigma)$ and $B(\Sigma)$ such that each induces either a triad or a fan, the attachments of $B(\Sigma)$ and $T(\Sigma)$ induce a 6-hole and there are no other adjacencies between the nodes of $T(\Sigma)$ and $B(\Sigma)$ (see Figure 4). $T(\Sigma)$ and $B(\Sigma)$ are the *sides* of Σ, $T(\Sigma)$ is the *top* and $B(\Sigma)$ the *bottom*.

Theorem 5.2 *A balanceable bipartite graph containing R_{10} as a proper induced subgraph has a biclique articulation.*

Theorem 5.3 *A balanceable bipartite graph that contains a connected 6-hole as an induced subgraph has an extended star cutset or a 6-join.*

Theorem 5.4 [7] *A balanceable bipartite graph not containing R_{10} or a connected 6-hole as induced subgraphs either is restricted balanceable or contains a 2-join or an extended star cutset.*

Now Theorem 5.1 follows from Theorems 5.2, 5.3 and 5.4.

6 Connection with Seymour's Decomposition of Totally Unimodular Matrices

Seymour [18] discovered a decomposition theorem for 0, 1 matrices that can be signed to be totally unimodular. The decompositions involved in his

theorem are 1-separations, 2-separations and 3-separations. A matrix B has a *k-separation* if its rows and columns can be partitioned so that

$$B = \begin{pmatrix} A^1 & D^2 \\ D^1 & A^2 \end{pmatrix}$$

where $r(D^1)+r(D^2) = k-1$ and the number of rows plus number of columns of A^i is at least k, for $i = 1,2$. (here $r(C)$ denotes the GF(2)-rank of 0,1 matrix C).

For a 1-separation $r(D^1) + r(D^2) = 0$. Thus both D^1 and D^2 are identically zero. The bipartite graph corresponding to the matrix B is disconnected.

For the 2-separation $r(D^1) + r(D^2) = 1$, thus w.l.o.g. D^2 has rank zero and is identically zero. Since $r(D^1) = 1$, after permutation of rows and columns, $D^1 = \begin{pmatrix} \mathbf{0} & E \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, where E is a matrix all of whose entries are 1. The 2-separation in the bipartite graph representation of B corresponds to a 1-join.

For the 3-separation $r(D^1) + r(D^2) = 2$. If both D^1 and D^2 have rank 1 then, after permutation of rows and columns,

$$D^1 = \begin{pmatrix} \mathbf{0} & E^1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad D^2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ E^2 & \mathbf{0} \end{pmatrix}$$

where E^1 and E^2 are matrices whose entries are all 1. This 3-separation in the bipartite graph representation of B corresponds to a 2-join.

When $r(D^1) = 2$ or $r(D^2) = 2$, it can be shown that the resulting 3-separation corresponds to a 2-join, a 6-join or to one of two other decompositions which each contain an extended star cutset.

In order to prove his decomposition theorem, Seymour used matroid theory. A matroid is *regular* if it is binary and its partial representations can be signed to be totally unimodular (see [21] for relevant definitions in matroid theory). The elementary families in Seymour's decomposition theorem consist of graphic matroids, cographic matroids and a 10-element matroid called

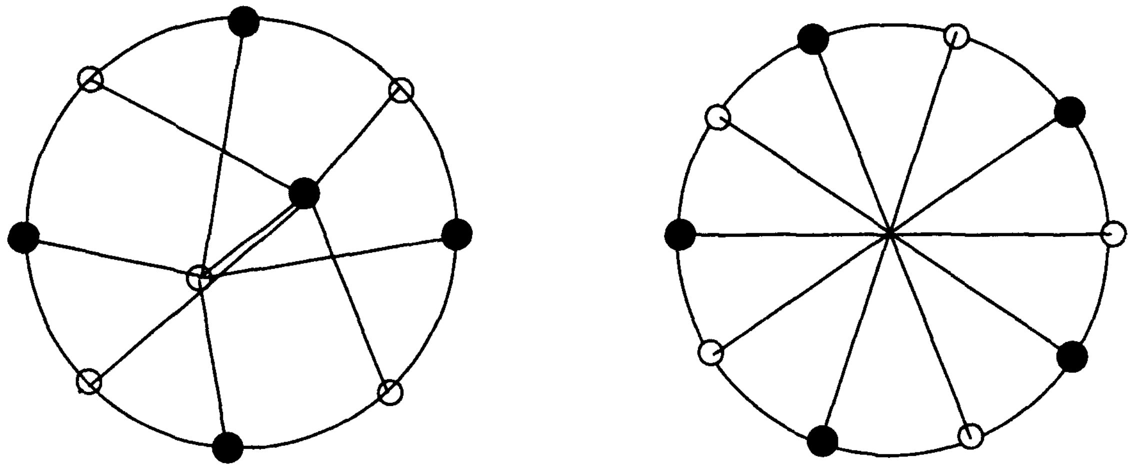


Figure 5: Representations of $\mathcal{R}_{10}$

$\mathcal{R}_{10}$. $\mathcal{R}_{10}$ has exactly two partial representations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The bipartite graph representations are shown in Figure 5.

Theorem 6.1 (Seymour [18]) *A regular matroid is either graphic, cographic, the 10-element matroid $\mathcal{R}_{10}$, or it contains a 1-, 2- or 3-separation.*

In order to prove Theorem 6.1, Seymour first showed that a regular matroid which is not graphic or cographic either contains a 1- or 2-separation or contains an $\mathcal{R}_{10}$ or an $\mathcal{R}_{12}$ minor, where $\mathcal{R}_{12}$ is a 12-element matroid having the following matrix as one of its partial representations.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Note that the bipartite graph representation of this matrix is a connected 6-hole where both sides are fans. So, this first part in Seymour's proof has some similarity with Theorem 5.4 stated above for balanceable bipartite graphs.

Then Seymour showed that, if a regular matroid contains an $\mathcal{R}_{10}$ minor, either it is $\mathcal{R}_{10}$ itself or it contains a 1-separation or a 2-separation. We show in Section 8 that if a balanceable bipartite graph contains an R_{10} as an induced subgraph, either it is R_{10} itself or it contains a biclique cutset.

Seymour completed his proof by showing that, for a regular matroid which contains an $\mathcal{R}_{12}$ minor, the 3-separation of $\mathcal{R}_{12}$ induces a 3-separation for the matroid. We show in Sections 9 - 13 that, for a balanceable bipartite graph which contains a connected 6-hole as an induced subgraph, either the 6-join of the connected 6-hole induces a 6-join of the whole graph or there is an extended star cutset.

Our proof differs significantly from Seymour's for the following reason. A regular matroid may have a large number of partial representations which lead to nonisomorphic bipartite graphs. This is the case for $\mathcal{R}_{12}$. All these partial representations are related through pivoting. In the case of $0,1$ balanceable matrices there is no underlying matroid, so pivoting cannot help reduce the number of cases. Since our proof is broken down differently from Seymour's, we do not consider all these cases explicitly either.

7 Definitions and Notation

Let G be a bipartite graph where the two sides of the bipartition are V^r and V^c. We say that G *contains* a graph Σ if Σ is an induced subgraph of G. A node $v \notin V(\Sigma)$ is *strongly adjacent* to Σ if $|N(v) \cap V(\Sigma)| \geq 2$. We say that a strongly adjacent node v is a *twin* of a node $x \in V(\Sigma)$ relative to Σ if $N(v) \cap V(\Sigma) = N(x) \cap V(\Sigma)$.

A *path* P is a sequence of distinct nodes $x_1, x_2, \ldots, x_n$, $n \geq 1$ such that $x_i x_{i+1}$ is an edge, for all $1 \leq i < n$. Let x_i and x_l be two nodes of P, where $l \geq i$. The path $x_i, x_{i+1}, \ldots, x_l$ is called the $x_i x_l$-subpath of P and is denoted by $P_{x_i x_l}$. We write $P = x_1, \ldots, x_{i-1}, P_{x_i x_l}, x_{l+1}, \ldots, x_n$ or $P = x_1, \ldots, x_i, P_{x_i x_l}, x_l, \ldots, x_n$. A cycle C is a sequence of nodes $x_1, x_2, \ldots, x_n, x_1$, $n \geq 3$, such that the nodes $x_1, x_2, \ldots, x_n$ form a path and $x_1 x_n$ is an edge. The node set of a path or a cycle Q is denoted by $V(Q)$.

Let A, B, C be three disjoint node sets such that no node of A is adjacent to a node of B. A path $P = x_1, x_2, \ldots, x_n$ *connects A and B* if one of the two endnodes of P is adjacent to at least one node in A and the other is adjacent to at least one node in B. The path P is a *direct connection between A and B* if, in the subgraph induced by the node set $V(P) \cup A \cup B$, no path connecting A and B is shorter than P. A direct connection P between A and B *avoids* C if $V(P) \cap C = \phi$. The direct connection P is said to be *from A to B* if x_1 is adjacent to some node in A and x_n to some node in B.

For $S \subseteq V(G)$, $N(S)$ denotes the set of nodes in $V(G) \setminus S$ which are adjacent to at least one node in S.

8 Splitter Theorem for R_{10}

An *extended* R_{10} is a bipartite graph induced by ten nonempty pairwise disjoint node sets $T_1, \ldots, T_{10}$ such that for every $1 \leq i \leq 10$, the node sets $T_i \cup T_{i-1}$, $T_i \cup T_{i+1}$ and $T_i \cup T_{i+5}$ all induce bicliques and these are the only edges in the graph. Throughout this section all the indices are taken modulo 10.

We consider a balanceable bipartite graph G which contains a node induced subgraph R isomorphic to R_{10}. We denote its node set by $\{1, \ldots, 10\}$ and for each $i = 1, \ldots, 10$, node i is adjacent to nodes $i-1, i+1$ and $i+5 \pmod{10}$.

The first step in the proof of the splitter theorem for R_{10} is to study the structure of the strongly adjacent nodes to R.

Theorem 8.1 *Let R be an R_{10} of G. If w is a strongly adjacent node to R, then w is a twin of a node in $V(R)$ relative to R.*

Proof: First assume that w has exactly two neighbors in R. If the neighbors of w in R are nodes 1 and 3, the hole $w, 1, 6, 7, 8, 3, w$ induces an odd wheel with center 2. If the neighbors of w in R are nodes 1 and 5, the hole $w, 1, 2, 7, 8, 9, 4, 5, w$ is an odd wheel with center 10. The other cases where w has two neighbors in R are isomorphic.

We now assume that node w is adjacent to at least three nodes in R. If node w is adjacent to nodes $i, i+2, i+4$, then there exists an odd wheel $i, i+1, i+2, i+3, i+4, i+5, i$ with center w. So w is adjacent to exact' three nodes $i, i+2, i+6$, showing that w is a twin of $i+1$. □

Definition 8.2 *Let R be an R_{10} of G. For $1 \le i \le 10$, let $T_i(R)$ be the set of nodes comprising node i in R and all the twins of node i relative to R. Let R^* be the graph induced by the node set $\cup_{i=1}^{10} T_i(R)$.*

Lemma 8.3 *R^* is an extended R_{10}.*

Proof: Let $u \in T_i(R)$ and $v \in T_j(R)$, where $1 \le i, j \le 10$. Let R' be the R_{10} obtained from R by substituting node u for node i. Now by Theorem 8.1, node v is twin of node j in R'. Hence nodes u and v are adjacent if and only if nodes i and j are adjacent. $\Box$

Theorem 8.4 *R^* satisfies the following two properties:*
(i) If node w is strongly adjacent to R^* then for some $1 \le i \le 10$, $N(w) \cap V(R^*) \subseteq T_i(R)$.
(ii) If R' is an R_{10} induced by the node set $\{x_1, \ldots, x_{10}\}$ where $x_i \in T_i(R)$ for $1 \le i \le 10$, then $T_i(R') = T_i(R)$.

Proof: To prove (i), assume that w is adjacent to $w_i \in T_i(R)$ and $w_j \in T_j(R)$, $i \ne j$. Let $R_{w_i w_j}$ be an R_{10} obtained from R by replacing node i with w_i and node j with w_j. Node w is now strongly adjacent to $R_{w_i w_j}$, so by Theorem 8.1 node w is a twin of a node in $R_{w_i w_j}$. Hence w is adjacent to a node k of R. Let R_{w_i} be an R_{10} obtained from R by replacing node i by w_i. Since w is adjacent to k and w_i, it is strongly adjacent to R_{w_i}, hence by Theorem 8.1 w is adjacent to a node $l \ne k$ of R. Now w is a strongly adjacent node of R and by Theorem 8.1 must be a twin of a node of R. Hence $w \in V(R^*)$, which contradicts our choice of w.

To prove (ii), note that Lemma 8.3 implies $T_i(R) \subseteq T_i(R')$, so it is enough to show that $T_i(R') \subseteq T_i(R)$. Let $u \in T_i(R')$ and suppose that $u \notin T_i(R)$. Then node u is strongly adjacent to R^* and by (i) we have a contradiction. $\Box$

Remark 8.5 *Considering Theorem 8.4 we can simplify the notation by replacing $T_i(R)$ by T_i.*

Definition 8.6 *For $1 \le i \le 10$, let K_i be the complete bipartite graph induced by the node set $T_{i-1} \cup T_i \cup T_{i+1} \cup T_{i+5}$.*

We now study the structure of paths between the nodes of R^*.

Lemma 8.7 *If $P = x_1, \ldots, x_n$ is a direct connection from T_i to $V(R^*) \setminus T_i$ in $G \setminus E(K_i)$, then the neighbors of x_n in R^* belong to a unique set T_j, where $j = i-1, i+1$ or $i+5$.*

Proof: Assume w.l.o.g. that x_1 is adjacent to node i. By Theorem 8.4 (i), $n > 1$ and node x_n has neighbors in exactly one T_j. Assume that for some $j \notin \{i-1, i+1, i+5\}$, x_n is adjacent to a node $v_j \in T_j$.

If $j = i+2$ then the hole $i, x_1, P, x_n, v_{i+2}, i+7, i+6, i+5, i$ induces an odd wheel with center $i+1$. If $j = i+3$ then the paths $P_1 = i, x_1, P, x_n, v_{i+3}$; $P_2 = i, i+1, i+2, v_{i+3}$ and $P_3 = i, i-1, i+4, v_{i+3}$ induce a $3PC(i, v_{i+3})$. If $j = i+4$ then the hole $i, x_1, P, x_n, v_{i+4}, i+3, i+8, i+7, i+6, i+1, i$ induces an odd wheel with center $i+2$. This completes the proof since the remaining cases are isomorphic to the above three. □

Lemma 8.8 *There cannot exist a path $P = x_1, \ldots, x_n$ with nodes belonging to $V(G) \setminus V(R^*)$ such that x_1 is adjacent to a node $v_i \in T_i$ and x_n is adjacent to a node $v_j \in T_j$, where $i \neq j$ and v_i and v_j are not adjacent.*

Proof: Let P be a shortest path contradicting the lemma. Hence P does not contain an intermediate node adjacent to a node in $T_i \cup T_j$. If no node x_l of P, $2 \leq l \leq n-1$, is adjacent to a node in $V(R^*)$ then P is a direct connection from T_i to $V(R^*) \setminus T_i$ in $G \setminus E(K_i)$ contradicting Lemma 8.7.

Let x_l be the node of P, with the smallest index, adjacent to a node in $V(R^*) \setminus (T_i \cup T_j)$, say x_l is adjacent to $w \in T_k$. By Lemma 8.7 and symmetry, we can assume w.l.o.g. that $k = i+1$ or $i+5$. No node in $V(R^*) \setminus T_k$ can be adjacent to an intermediate node of P, otherwise P is not a shortest path contradicting the lemma. Let x_m be the node of P with highest index which is adjacent to a node $v_k \in T_k$.

Case 1: k=i+1. Lemma 8.7 applied to $x_m, \ldots, x_n$ and the minimality of P show that $j = i+2$ or $i+6$.

Cases 1.1: j=i+2. Let $H_1 = v_i, x_1, P, x_n, v_{i+2}, i+3, i+4, i-1, v_i$ and $H_2 = v_i, x_1, P, x_n, v_{i+2}, i+7, i+6, i+5, v_i$. Now either H_1 or H_2 induces an odd wheel with center $i+1$.

Case 1.2: j=i+6. The hole $v_i, x_1, P, x_n, v_{i+6}, i+7, i+2, i+3, i+4, i-1, v_i$ induces an odd wheel with center $i+5$.

Case 2: k = i+5. Lemma 8.7 applied to $x_m, \ldots, x_n$ and the minimality of P show that $j = i + 4$. Now the hole $v_i, x_1, P, x_n, v_{i+4}, i+3, i+8, i+7, i+6, i+1, v_i$ induces an odd wheel with center $i + 2$.

□

Theorem 8.9 *If a balanceable bipartite graph G contains R_{10} then either G is R_{10} itself or G contains a biclique cutset.*

Proof: Let R be an R_{10} of G. By Lemma 8.3, R^* is an extended R_{10}. Assume that $V(G) \neq V(R^*)$. Let w be a node in $V(G) \setminus V(R^*)$ adjacent to a node in T_l. If the biclique K_l is not a cutset of G, seperating w from $V(R^*)$, then a path contradicting Lemma 8.8 exists. Hence $V(G) = V(R^*)$. If G is not R_{10}, then at least one of the node sets $T_i(R)$ has cardinality greater than one. W.l.o.g. let u and v be two nodes in $T_1(R)$. Now $\{u\} \cup N(u)$ is a star cutset seperating v from the rest of the graph. □

9 Decomposition of Connected Six-holes

In the remaining sections, we assume that G is a balanceable bipartite graph and Σ is a connected 6-hole induced by $T(\Sigma)$ and $B(\Sigma)$. We prove that either G contains an extended star cutset or it has a 6-join which separates the top and the bottom of Σ.

We denote by $H = h_1, h_2, h_3, h_4, h_5, h_6, h_1$ the 6-hole of Σ induced by the attachments of $T(\Sigma)$ and $B(\Sigma)$ and we assume that $h_1, h_3, h_5 \in T(\Sigma)$ and $h_2, h_4, h_6 \in B(\Sigma)$. We also assume $h_1, h_3, h_5 \in V^c$ and $h_2, h_4, h_6 \in V^r$. It will be convenient to define the index of h_j modulo 6. If $T(\Sigma)$ is a triad, the three paths defining it are denoted by P_1, P_2 and P_3 and the meet is denoted by t. For connected 6-hole Σ', Σ'' and Σ^k, we denote the respective 6-holes by $H' = h'_1, h'_2, h'_3, h'_4, h'_5, h'_6, h'_1$, $H'' = h''_1, h''_2, h''_3, h''_4, h''_5, h''_6, h''_1$, and $H^k = h^k_1, h^k_2, h^k_3, h^k_4, h^k_5, h^k_6, h^k_1$.

Remark 9.1 *Let X be one of the sides of a balanceable connected 6-hole Σ. If X is a triad, its meet belongs to the same side of the bipartition as its attachments, else X contains a 3-path configuration. If X is a fan, its center has a positive even number of neighbors on the path of the fan connecting the other two attachments, else X contains an odd wheel. Hence X cannot be both a triad and a fan.*

Remark 9.2 *Let h_i and h_j be two distinct attachments of a side X of Σ. There is a unique chordless path in X, connecting h_i and h_j. This path is denoted by P_{ij}. For any pair of nodes x and y in $V(\Sigma)$, there exists a hole containing x and y whose node set is included in $V(\Sigma)$.*

We use the following theorems, proved in [7] Part VI, about the structure of strongly adjacent nodes to an even wheel. We first introduce the relevant notation. Two sectors of a wheel are *adjacent* if they have a common endnode. A *bicoloring of a wheel* is an assignment of colors to the intermediate nodes of its sectors so that the nodes in the same sector have the same color and nodes of adjacent sectors have distinct colors. The endnodes of sectors are left unpainted. Note that a wheel is bicolorable if and only if it is even.

Theorem 9.3 *Let (W, v), $v \in V^r$, be an even wheel in a balanceable bipartite graph, and let $u \in V^c \backslash N(v)$ be a node with neighbors in at least two distinct sectors of the wheel (W, v). Then u satisfies one of the following properties:*

Type a *Node u has exactly two neighbors in W and these neighbors belong to two distinct sectors having the same color.*

Type b *There exists one sector, say S_j with endnodes v_i and v_k, such that u has a positive even number of neighbors in S_j and has exactly two neighbors in $V(W) \setminus V(S_j)$, adjacent to v_i and v_k respectively.*

Theorem 9.4 *Let (W, v), $v \in V^r$, be an even wheel in a balanceable bipartite graph, and let $u \in V^c \cap N(v)$ be a node which is strongly adjacent to (W, v). Then u satisfies one of the following properties:*

Type a *Node u has exactly one neighbor in W.*

Type b *Node u is not of Type a and in each sector of (W, v), u has either 0 or an odd number of neighbors. It follows that u has neighbors in an even number of sectors and that the number of consecutive sectors without neighbors of u, between two sectors with neighbors of u, is even.*

Classification 9.5 *A node $u \in V^r$, strongly adjacent to an even wheel (W, v), $v \in V^r$, is classified as follows:*

Type a *There exists a sector of (W, v) containing all the nodes of $N(u) \cap W$.*

Type b *Node u is not of Type a and all its neighbors in W are unpainted. Note that, in particular, the center v of the wheel is of Type b.*

Type c *Node u is not of Types a or b and all its painted neighbors in W have the same color.*

Type d *Node u has painted neighbors of both colors.*

10 Strongly Adjacent Nodes to a Connected 6-Hole

The first step in our decomposition of a connected 6-hole Σ is the study of the strongly adjacent nodes. We use notation introduced in Section 9.

Lemma 10.1 *If $T(\Sigma)$ is a triad and w is adjacent to its meet t, then all nodes of $N(w) \cap T(\Sigma)$ are contained in a unique path P_j of the triad, where $j = 1, 3$ or 5.*

Proof: Assume not. Then w.l.o.g. w has neighbors in $P_1 \setminus \{t\}$ and $P_3 \setminus \{t\}$. Since the hole h_2, P_1, P_3, h_2 induces a wheel with center w, the node w has a positive even number of neighbors in one of the paths P_1, P_3 and an odd number (greater than one) of neighbors in the other. Let $H_1 = h_6, P_1, P_5, h_6$ and $H_2 = h_4, P_3, P_5, h_4$. Now either (H_1, w) or (H_2, w) induces an odd wheel. $\square$

10.1 Strongly Adjacent Nodes Having Neighbors Both in $T(\Sigma)$ and $B(\Sigma)$

In this section, w denotes a strongly adjacent node to Σ and we assume w.l.o.g. that w is in V^r.

Theorem 10.2 *If $w \in V^r$ has neighbors both in $T(\Sigma)$ and $B(\Sigma)$, then $N(w) \cap T(\Sigma) = \{h_i, h_j\}$, $i \neq j$, $i, j = 1, 3$ or 5.*

To prove this theorem, we need the following lemmas:

Lemma 10.3 *If $T(\Sigma)$ is a triad and $w \in V^r$ has neighbors both in $T(\Sigma)$ and $B(\Sigma)$, then $N(w) \cap T(\Sigma) = \{h_i, h_j\}$, $i \neq j$, where $i, j = 1, 3$ or 5.*

Proof: First we show that the neighbors of w in $T(\Sigma)$ cannot all be contained in the same path of $T(\Sigma)$. Assume the contrary i.e. assume that for some $j = 1, 3$ or 5, $N(w) \cap T(\Sigma) \subseteq P_j$. Then since $w \in V^r$ is not adjacent to $h_2, h_4, h_6 \in V^r$ but has at least one neighbor in $B(\Sigma)$, there is a $3PC(t, h_{j+3})$ where t is the meet of $T(\Sigma)$. Node w is not adjacent to the meet t, since otherwise by Lemma 10.1 the neighbors of w in $T(\Sigma)$ would all be contained in the same path of $T(\Sigma)$. Then node w has neighbors in at most two paths of $T(\Sigma)$, since otherwise there is a $3PC(w, t)$. Therefore node w has neighbors in exactly two distinct paths of $T(\Sigma)$, say P_1 and P_3. Let $w_1 \in P_1$ and $w_3 \in P_3$ be neighbors of w. Assume w.l.o.g. that $w_3 \neq h_3$. Now there is a $3PC(w, t)$ where the intermediate nodes of the three paths are included respectively in $V(P_1), V(P_3)$ and $V(P_5) \cup (B(\Sigma) \setminus \{h_2, h_6\})$. Therefore $N(w) \cap T(\Sigma) = \{h_1, h_3\}$. □

We now study the case where $T(\Sigma)$ is a fan and we assume w.l.o.g. that h_3 is the center node of the fan.

Lemma 10.4 *If $T(\Sigma)$ is a fan and $w \in V^r$ has neighbors both in $T(\Sigma)$ and $B(\Sigma)$ but w is not adjacent to h_3, then $N(w) \cap T(\Sigma) = \{h_1, h_5\}$.*

Proof: Let H_{15} be the hole induced by the paths P_{15} in $T(\Sigma)$ and P_{24} in $B(\Sigma)$. We first show the following claim:

Claim 1: Node w has more than one neighbor in $T(\Sigma)$.

Proof of Claim 1: Assume not and let w_1 be the unique neighbor of w in $T(\Sigma)$. If w_1 belongs to a sector of (H_{15}, h_3) having either h_2 or h_4 as endnode, there is an odd wheel with center h_3. Otherwise there is a $3PC(w_1, h_6)$. This proves Claim 1.

So w is not adjacent to h_3 and is strongly adjacent to H_{15} and therefore w is of Type a or b[9.3] relative to (H_{15}, h_3).

If w is of Type a[9.3] with neighbors w_1 and w_2 in H_{15}, Claim 1 shows $w_1, w_2 \in T(\Sigma)$. Hence w has a neighbor in $B(\Sigma) \setminus V(H_{15})$. Now w_1, w_2 must coincide with h_1, h_5, else there is a $3PC(w, h_3)$.

So w is of Type b[9.3]. If all the neighbors of w in H_{15} belong to $T(\Sigma)$, there is a $3PC(w, h_3)$. If all but one of the neighbors of w belong to $T(\Sigma)$, there is an odd wheel with center w. The structure of a Type b[9.3] node shows that the only remaining possibility is that the neighbors in $T(\Sigma)$ of w are h_1, h_5, completing the proof of the lemma. □

Lemma 10.5 *If $T(\Sigma)$ is a fan, $w \in V^r$ has neighbors both in $T(\Sigma)$ and $B(\Sigma)$ and w is adjacent to h_3, then $N(w) \cap T(\Sigma) = \{h_1, h_3\}$ or $\{h_3, h_5\}$.*

Proof: If w has no neighbor in $T(\Sigma) \setminus \{h_3\}$ then, since w has a neighbor in $B(\Sigma)$, there is a $3PC(h_3, h_6)$.

So w is strongly adjacent to (H_{15}, h_3) and satisfies Theorem 9.4, where H_{15} denotes the hole induced by the paths P_{15} in $T(\Sigma)$ and P_{24} in $B(\Sigma)$. We first show that w has a unique neighbor in $T(\Sigma) \setminus \{h_3\}$.

This is clearly the case if node w is of Type a[9.4], so assume node w is of Type b[9.4]. If w is adjacent to a node in the sector $B(\Sigma) \cap V(H_{15})$ of (H_{15}, h_3), then Theorem 9.4 shows that w has an odd number of neighbors in $T(\Sigma) \setminus \{h_3\}$. Hence w has exactly one neighbor in $T(\Sigma) \setminus \{h_3\}$, else this node set together with node h_6 induces an odd wheel with center w. If w is not adjacent to $B(\Sigma) \cap V(H_{15})$ and it has a unique neighbor w_1 in $B(\Sigma) \setminus V(H_{15})$, then there is a $3PC(w_1, h_2)$ or a $3PC(w_1, h_4)$. Finally, if w is not adjacent to $B(\Sigma) \cap V(H_{15})$ and it has at least two neighbors in $B(\Sigma) \setminus V(H_{15})$, then there is a $3PC(w, h_1)$ or a $3PC(w, h_5)$.

Let w_1 be the unique neighbor of w in $T(\Sigma) \setminus \{h_3\}$. If w_1 is distinct from h_1 and h_5, then there is a $3PC(w_1, h_6)$. □

Proof of Theorem 10.2: The proof of the theorem follows from Lemmas 10.3, 10.4 and 10.5. □

10.2 Strongly Adjacent Nodes Having Neighbors Only in One Side of Σ

In this section we assume w.l.o.g. that the strongly adjacent node w has no neighbor in $B(\Sigma)$.

Theorem 10.6 *If $T(\Sigma)$ is a triad, then w is one of the following types, see Figure 6 :*

Type a *$N(w) \cap T(\Sigma) \subset V(P_i)$ for $i = 1, 3$ or 5.*

Type b *$w \in V^c$ has at least one neighbor in each path P_1, P_3 and P_5.*

Type c *$w \in V^c$ has neighbors in exactly two of the paths P_1, P_3 and P_5. Furthermore w either has an even number of neighbors in each of the two paths or has one neighbor in each path and both neighbors are adjacent to the meet t.*

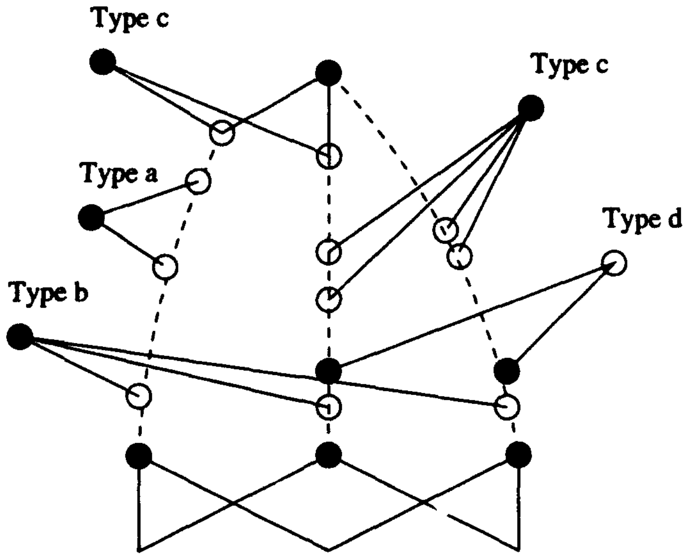


Figure 6: Strongly adjacent nodes with all neighbors in a triad $T(\Sigma)$

Type d *$w \in V^r$ is not adjacent to the meet t and has two neighbors in $T(\Sigma)$ which belong to distinct paths of $T(\Sigma)$.*

Proof: If some path of $T(\Sigma)$ contains all the nodes in $N(w) \cap T(\Sigma)$, then we have Type a. If $w \in V^c$ has neighbors in all three paths, we have Type b.

Assume now that $w \in V^c$ has neighbors in exactly two paths, say P_1 and P_3. Then w cannot have an even number of neighbors in one path and an odd number in the other, else there is an odd wheel with center w. If w has an odd number, greater than one, of neighbors in one of the paths, then P_5 closes an odd wheel with center w. Let w_1 be the unique neighbor of w in P_1 and let w_2 be the unique neighbor of w in P_3. Then w_1 is adjacent to t, else there is a $3PC(w_1, t)$. Similarily w_2 is adjacent to t. This yields Type c.

Finally assume that $w \in V^r$ is not of Type a. Lemma 10.1 shows that w is not adjacent to t. If w has neighbors in all three paths of $T(\Sigma)$, there is a $3PC(w, t)$. So w has neighbors in exactly two paths and if w has more than two neighbors in $T(\Sigma)$ there is a $3PC(w, t)$. $\square$

Remark 10.7 *Let Σ be a connected 6-hole whose top is a fan with center h_3 and let H_{15} be the hole induced by paths P_{15} and P_{24}. A node w strongly*

adjacent to Σ but with no neighbor in $B(\Sigma)$ can be of any of the types described in Theorems 9.3, 9.4, 9.5 relative to (H_{15}, h_3).

Theorem 10.8 *If w is a strongly adjacent node to Σ, with no neighbor in $B(\Sigma)$, then either w belongs to a connected 6-hole with top contained in $T(\Sigma) \cup \{w\}$, bottom $B(\Sigma)$ and 6-hole H or one of the following holds:*

- *$T(\Sigma)$ is a triad and w is of Type c[10.6] with exactly two neighbors in $T(\Sigma)$.*
- *$T(\Sigma)$ is a triad and w is of Type d[10.6] adjacent to two nodes of the 6-hole.*
- *$T(\Sigma)$ is a fan, say with center h_3, and w is of Type a[9.4] relative to (H_{15}, h_3).*

Proof: If $T(\Sigma)$ is a triad, the proof follows from Theorem 10.6 by inspection. Now assume $T(\Sigma)$ is a fan with center h_3 and let H_{15} be the hole induced by paths P_{15} and P_{24}. If w is adjacent to h_3, then w is strongly adjacent to the wheel (H_{15}, h_3) and the theorem follows from Remark 10.7. If w is not adjacent to h_3, let Q be the shortest path between h_1 and h_5 containing w, in $T(\Sigma) \cup \{w\} \setminus \{h_3\}$.

If h_3 is adjacent to a node of Q, then $V(Q) \cup \{h_3\}$ induces a fan with attachments h_1, h_3, h_5.

If h_3 is not adjacent to a node of Q, let R be a direct connection between h_3 and $V(Q)$, using nodes of $T(\Sigma)$. Then $V(Q) \cup V(R) \cup \{h_3\}$ induces a triad with attachments h_1, h_3 and h_5 which, together with $B(\Sigma)$, induces a connected 6-hole. □

Classification 10.9 *Theorems 10.2 and 10.8 partition the strongly adjacent nodes w to Σ into the following classes:*

Type a *Node w belongs to a connected 6-hole with nodes in $V(\Sigma) \cup \{w\}$.*

Type b *Node w is adjacent to exactly two nodes of Σ and these two nodes belong to the 6-hole of Σ. Such a node w is called a fork.*

Type c *Node w has exactly two neighbors in Σ, both belonging to the same side which is a triad and both neighbors are adjacent to the meet of the triad.*

Type d *Node w has exactly two neighbors in Σ, both belonging to the same side which is a fan, say with center h_i, and w is adjacent to h_i and one other node which is not an attachment of the fan.*

11 Direct Connections from Top to Bottom

Lemma 11.1 *Every direct connection $P = x_1, \ldots, x_n$ between $T(\Sigma)$ and $B(\Sigma)$ in $G \setminus E(H)$ is of one of the following types:*

- *$n = 1$ and x_1 is a strongly adjacent node satisfying Theorem 10.2.*
- *One endnode of P is a fork, adjacent to h_{i-1} and h_{i+1} and the other endnode of P is adjacent to a node of $V(\Sigma) \setminus V(H)$*
- **Bridge of Type a** *Nodes x_1 and x_n are not strongly adjacent to Σ and their unique neighbors in Σ are two adjacent nodes of the 6-hole of Σ.*

  Bridge of Type b1 *One endnode of P is a fork, say x_1 is adjacent to h_1 and h_3, and x_n has a unique neighbor in Σ which is h_2.*

  Bridge of Type c1 *Node x_1 is a fork, say adjacent to h_1 and h_3, and x_n is also a fork, adjacent to h_2 and either h_4 or h_6.*

Proof: If $n = 1$, x_1 is a strongly adjacent node with neighbors both in $T(\Sigma)$ and $B(\Sigma)$ and this possibility is described in Theorem 10.2. So we assume $n > 1$, x_1 has no neighbors in $B(\Sigma)$ and x_n has no neighbors in $T(\Sigma)$.

Case 1: Neither x_1 nor x_n is a fork of Σ.

Case 1.1: Nodes x_1 and x_n are either not strongly adjacent to Σ or they are of Type a[10.9].

Assume x_1 is a strongly adjacent node. Let Σ' be a connected 6-hole containing x_1 and having node set included in $V(\Sigma) \cup \{x_1\}$. Node x_1 does not belong to the 6-hole of Σ', since x_1 has no neighbor in $B(\Sigma)$. This shows $n > 2$, otherwise x_n is a strongly adjacent node with neighbors both in the top and bottom of Σ', and since x_n is not a fork of Σ this violates Theorem 10.2. Therefore, after possibly modifying P and Σ appropriately, we can assume w.l.o.g. that both x_1 and x_n are not strongly adjacent to Σ. Let y and z be the unique neighbors of x_1 and x_n in Σ, respectively.

If y and z belong to the same side of the bipartition, assume w.l.o.g. that $y \in V^r$, $y \in V(P_{15})$ and y is not adjacent to h_1. There exists a $3PC(h_1, y)$ using P and the hole induced by $V(P_{15}) \cup \{h_6\}$, unless z coincides with h_4 or h_6. Assume $z = h_6$. Then there is a $3PC(h_3, h_6)$ unless y is adjacent to h_5. But then there is an odd wheel with center h_5 and hole induced by $V(P) \cup V(P_{35} \setminus \{h_5\}) \cup V(P_{46})$. Assume $z = h_4$. Then there is a $3PC(y, h_5)$ unless y is adjacent to h_5. But then there is an odd wheel with center h_5 and hole induced by $V(P) \cup V(P_{15} \setminus \{h_5\}) \cup V(P_{46})$.

By Remark 9.2, y and z belong to a hole H with node set included in $V(\Sigma)$. If y and z belong to opposite sides of the bipartition and they are not adjacent, the path P together with H induces a $3PC(y, z)$. If y and z are adjacent, then they belong to the 6-hole of Σ and P is a bridge of Type a. Note that, in this case, P and Σ were not modified.

Case 1.2: Node x_1 is of Type c[10.9].

Then $T(\Sigma)$ is a triad. Assume w.l.o.g. that the neighbors of x_1 belong to the paths P_1 and P_3. If x_n is adjacent to a node in $B(\Sigma) \setminus \{h_4, h_6\}$, there is a $3PC(x_1, h_2)$. If x_n is adjacent to h_6 only, there is a $3PC(x_1, h_6)$. If x_n is adjacent to h_4 only, there is a $3PC(x_1, h_4)$. Since x_n is not a fork, Case 1.2 cannot occur.

Case 1.3: Node x_1 is of Type d[10.9].

Then $T(\Sigma)$ is a fan, say with center h_3 and x_1 is adjacent to h_3 and one other node of the fan, say y, distinct from h_1 and h_5. If x_n is adjacent to a node in $B(\Sigma) \setminus \{h_2, h_4\}$, there is a $3PC(y, h_6)$. If x_n is adjacent to h_2 only, there is a $3PC(y, h_2)$. If x_n is adjacent to h_4 only, there is a $3PC(y, h_4)$. Since x_n is not a fork, Case 1.3 cannot occur.

Case 2: Either x_1 or x_n is a fork of Σ, but not both.

W.l.o.g. assume x_1 is a fork adjacent to h_1 and h_3. If x_n is not adjacent to a node of $V(\Sigma) \setminus V(H)$ then x_n has a unique neighbor y in Σ, where $y = h_2, h_4$ or h_6. If $y = h_2$, we have a bridge of Type b1. If $y = h_4$ or h_6, say h_4, the hole induced by $V(P) \cup V(P_{24}) \cup \{h_1\}$ forms an odd wheel with center h_3.

Case 3: Both x_1 an x_n are forks of Σ.

We have a bridge of Type c1, unless x_1 is adjacent to, say h_1 and h_3, and x_n is adjacent to h_4 and h_6. But, in this case, there is a $3PC(x_1, x_n)$ if x_1 is not adjacent to x_n, and an odd wheel with center x_1 if x_1 is adjacent to x_n. □

Lemma 11.2 *Every direct connection $P = x_1,..,x_n$ from $T(\Sigma) \setminus \{h_1\}$ to $B(\Sigma)$ avoiding $\{h_1\}$ in $G \setminus E(H)$ is either one of the types described in Lemma 11.1 or $n > 1$, there exists a node x_i, $1 < i < n$, adjacent to h_1 and P satisfies one of the following alternatives:*

- *Node x_1 is adjacent to at least one node in $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ and x_n is a fork adjacent to h_2 and h_6.*
- **Bridge of Type b2** *Node x_n is adjacent either to h_2 or h_6, say h_2 and to no other node of Σ. Node x_1 is adjacent to h_3, possibly h_1 and to no other node of Σ.*

  Bridge of Type c2 *Node x_n is a fork adjacent to h_2 and h_6. Node x_1 is adjacent to either h_3 or h_5 but not both, possibly h_1 and to no other node of Σ.*

Proof: If no node x_i, $1 < i < n$, is adjacent to h_1, then P is also a direct connection from $T(\Sigma)$ to $B(\Sigma)$ in $G \setminus E(H)$. Hence P satisfies Lemma 11.1.

Let x_j, $1 < j < n$, be the node of highest index which is adjacent to h_1. The subpath $P_{x_j x_n}$ of P is a direct connection from $T(\Sigma)$ to $B(\Sigma)$ satisfying Lemma 11.1. Since x_j is adjacent to h_1 only, $P_{x_j x_n}$ is a bridge of Type a or b [11.1].

Assume that $P_{x_j x_n}$ is a bridge of Type a [11.1]. Then x_n is adjacent to either h_2 or h_6, say h_2. If x_1 has a neighbor in $T(\Sigma) \setminus \{h_1, h_3\}$, then there exists a chordless path Q connecting x_1 to h_5 whose intermediate nodes belong to $T(\Sigma) \setminus \{h_1, h_3\}$. Now one of the two holes formed by the nodes of P, Q and either P_{26} or P_{24} contains an odd number of neighbors of h_1. So the neighbors of x_1 in $T(\Sigma)$ are contained in $\{h_1, h_3\}$ and x_1 is adjacent to h_3. This yields a bridge of Type b2 [11.2].

Assume now that $P_{x_j x_n}$ is a bridge of Type b1 [11.1]. Then x_n is adjacent to h_2 and h_6. If x_1 has a neighbor in $T(\Sigma) \setminus \{h_1, h_3, h_5\}$, then the first possibility of Lemma 11.2 holds. So the neighbors of x_1 in $T(\Sigma)$ are contained in $\{h_1, h_3, h_5\}$. If x_1 is a fork, adjacent to h_3 and h_5, then there is a $3PC(x_1, x_n)$. This yields a bridge of Type c2 [11.2]. □

Lemma 11.3 *Every direct connection $P = x_1, \ldots, x_n$ from $T(\Sigma) \setminus \{h_1, h_3\}$ to $B(\Sigma)$ avoiding $\{h_1, h_3\}$ in $G \setminus E(H)$ is either described in Lemma 11.2, or P satisfies the following two conditions:*

- *There exist nodes x_j and x_k, $1 < j, k < n$, of P adjacent to h_1 and h_3 respectively (possibly $j = k$).*

- *Let x_j, $j < n$ be the node of highest index adjacent to h_1 or h_3, say h_i. Then x_n is a fork of Σ adjacent to h_{i-1} and h_{i+1}.*

Proof: If no node x_j of P, $1 < j < n$ is adjacent to h_3, then P is also a direct connection from $T(\Sigma) \setminus \{h_1\}$ to $B(\Sigma)$ avoiding $\{h_1\}$ in $G \setminus E(H)$ and is described in Lemma 11.2. By symmetry, a similar conclusion holds if no node x_j, $1 < j < n$ is adjacent to h_1. Hence the first condition of the lemma holds. Let x_j be the node of highest index adjacent to h_1 or h_3, say h_3, such that there exists at least one x_k, $k \geq j$ adjacent to h_1 but no node x_l, $l > j$ is adjacent to h_3. Then the subpath $P_{x_j x_n}$ of P is a direct connection from $T(\Sigma) \setminus \{h_1\}$ to $B(\Sigma)$ avoiding $\{h_1\}$ in $G \setminus E(H)$.

Claim 1: Node x_n has at most two neighbors in Σ, which are h_2 and possibly either h_4 or h_6.

Proof of Claim 1: Let x_l be a node of P adjacent to h_1 and having highest index. (Obviously $l \geq j$). Then the subpath $P_{x_l x_n}$ of $P_{x_j x_n}$ is a direct connection from $T(\Sigma)$ to $B(\Sigma)$ in $G \setminus E(H)$. If $l > j$ then x_l has h_1 as unique neighbor in Σ and by Lemma 11.1 the claim holds. If $l = j$ then the neighbors of x_l in Σ are h_1 and h_3 and by Lemma 11.1, $P_{x_l x_n}$ is either a bridge of Type b1 or c1, in which case the claim holds, or x_n has a neighbor in $B(\Sigma) \setminus \{h_2, h_4, h_6\}$. Let P' be a direct connection using nodes of $B(\Sigma)$ between x_n and h_6 and avoiding $\{h_2, h_4\}$ and P'' be a direct connection using nodes of $B(\Sigma)$ between x_n and h_4 and avoiding $\{h_2, h_6\}$. Let P^* be a direct connection using nodes of $T(\Sigma)$ between x_1 and h_5 and avoiding $\{h_1, h_3\}$ and consider the holes $H' = x_1, P^*, h_5, h_6, P', x_n, P, x_1$ and $H'' = x_1, P^*, h_5, h_4, P'', x_n, P, x_1$. Then if P has more than one neighbor of h_1, either (H', h_1) or (H'', h_1) is an odd wheel. Otherwise, if h_1 has a unique neighbor, say h^* in P, there is a $3PC(h^*, h_5)$. This completes the proof of Claim 1.

Finally assume that x_n has h_2 as unique neighbor in Σ. Let Q be a direct connection between h_5 and h_2 avoiding $\{h_1, h_3\}$ and using nodes of $T(\Sigma) \cup V(P)$ and let $C' = h_5, Q, h_2, P_{26}, h_6, h_5$, $C'' = h_5, Q, h_2, P_{24}, h_4, h_5$. Then either (C', h_1) or (C'', h_1) is an odd wheel. $\square$

12 Expanding the 6-Hole

Definition 12.1 *A fork z_i of Σ adjacent to nodes h_{i-1} and h_{i+1}, i odd, is attached in Σ if there exists a direct connection $P = x_1, \ldots, x_n, z_i$ from $T(\Sigma) \setminus \{h_i, h_j\}$ to $B(\Sigma)$ avoiding $\{h_i, h_j\}$ in $G \setminus E(H)$ where $j = i-2$ or $i+2$ and x_1 is adjacent to at least one node in $T(\Sigma) \setminus \{h_1, h_3, h_5\}$. The path $x_1, \ldots, x_n$ is an attachment of z_i to Σ. If i is even, an attached fork is defined accordingly.*

We will maintain the convention that if $P = x_1, \ldots, x_n$ is an attachment of z_i to Σ, then x_n is adjacent to z_i and x_1 to at least one node in $\Sigma \setminus \{h_1, \ldots, h_6\}$.

Definition 12.2 *Let $H_i(\Sigma)$ be the set of attached forks, adjacent to h_{i-1} and h_{i+1} together with the nodes adjacent to h_{i-1} and h_{i+1} and having neighbors in both sides of Σ. Note that $h_i \in H_i(\Sigma)$.*

Lemma 12.3 *For every node $z_i \in H_i(\Sigma)$, say i odd, there exists a connected 6-hole Σ' having the following properties:*

- *$B(\Sigma') = B(\Sigma)$*
- *If z_i is a strongly adjacent node, with neighbors in both sides of Σ, $T(\Sigma') \subseteq (T(\Sigma) \cup \{z_i\}) \setminus \{h_i\}$.*

  If z_i is a fork of Σ, with attachment P, $T(\Sigma') \subseteq (T(\Sigma) \cup \{z_i\} \cup V(P)) \setminus \{h_i\}$ and $V(P) \subseteq T(\Sigma')$.

Proof: Assume w.l.o.g. $i = 1$ and let P_{35} be the path of $T(\Sigma)$ connecting h_3 and h_5.

Case 1: z_1 is a strongly adjacent node with neighbors in top and bottom of Σ.

If z_1 is adjacent to a node in P_{35}, then $\{z_1\} \cup V(P_{35})$ induces a fan top of Σ'. (Hence z_1 has more than one neighbor in P_{35}). If z_1 is adjacent to a node in $T(\Sigma) \setminus V(P_{35})$ but to no node in P_{35}, let Q be a direct connection in $T(\Sigma)$ between z_1 and $V(P_{35}) \setminus \{h_3, h_5\}$ avoiding $\{h_1, h_3, h_5\}$. First note that the endnode of Q adjacent to P_{35} has a unique neighbor in $V(P_{35})$. Now if $V(Q) \cup V(P_{35}) \cup \{z_1\}$ does not induce a triad then either h_3 or h_5 must have a neighbor in Q. By construction not both h_3 and h_5 have a neighbor in Q. Hence Σ' has a fan top with center h_3 or h_5.

Case 2: Node z_1 is a fork of Σ, with attachment $P = x_1, \ldots, x_m$.

Case 2.1: Either h_3 or h_5 is adjacent to a node in $V(P) \setminus \{x_1\}$.

Assume w.l.o.g. that h_3 is adjacent to a node in $V(P) \setminus \{x_1\}$. Let R be a direct connection in $T(\Sigma) \cup V(P) \cup \{z_1\}$ between z_1 and h_5 avoiding $\{h_1, h_3\}$. Then $V(P) \subseteq V(R)$, hence h_3 has a neighbor in R, and so R induces a fan top with center h_3.

Case 2.2: Neither h_3 nor h_5 is adjacent to a node in $V(P) \setminus \{x_1\}$.

Then the path induced by the node set $\{z_1\} \cup V(P)$ satisfies either the second alternative of Lemma 11.1 or the first alternative of Lemma 11.2. Assume first that x_1 has a neighbor in $V(P_{35})$. If x_1 is strongly adjacent to P_{35}, we can shorten P and modify P_{35} accordingly. If z_1 becomes adjacent to a node in $V(P_{35})$ the argument of Case 1 holds. Now consider the case where x_1 has a unique neighbor y in P_{35}. If y is adjacent to h_3 or h_5, there is an odd wheel with center h_3 or h_5. Otherwise $V(P_{35}) \cup V(P) \cup \{z_1\}$ induces a triad.

If x_1 has no neighbors in P_{35}, let Q be a direct connection in $T(\Sigma) \cup V(P)$ between z_1 and $V(P_{35}) \setminus \{h_3, h_5\}$ avoiding $\{h_1, h_3, h_5\}$. Then by construction $V(P) \subseteq V(Q)$ and Q cannot have both a neighbor of h_3 and h_5. Hence $V(P_{35}) \cup V(Q)$ induces a fan or a triad top. $\square$

Definition 12.4 *A connected 6-hole Σ' satisfying Lemma 12.3 is said to be obtained from Σ by substituting node z_i (with attachment P_{z_i}) for h_i. If i is even, $T(\Sigma) = T(\Sigma')$ and z_i is said to be substituted in the bottom. If i is odd, $B(\Sigma) = B(\Sigma')$ and z_i is said to be substituted in the top.*

Lemma 12.5 *Let Σ' be a connected 6-hole obtained from Σ by substituting node $z_i \in H_i(\Sigma)$ for h_i. Then $H_j(\Sigma) = H_j(\Sigma')$ for $j \in \{i-1, i+1, i+3\}$.*

Proof: Assume w.l.o.g. $i = 1$. Let $z_1 \in H_1(\Sigma)$, and if z_1 is a fork of Σ, let $P_{z_1} = x_1, \ldots, x_n$ be an attachment of z_1 to Σ. Assume w.l.o.g. that h_5 is not adjacent to a node of $V(P_{z_1}) \setminus \{x_1\}$. Let $z_j \in H_j(\Sigma)$, where j is even. If z_j is a fork of Σ, let $P_{z_j} = y_1, \ldots, y_m$ be an attachment of z_j to Σ. Let Σ'' be the connected 6-hole obtained from Σ by substituting node z_j (with attachment P_{z_j}) for node h_j.

Claim 1: No node x_k, $1 \leq k \leq n$, is adjacent to or coincident with a node in $V(P_{z_j}) \cup \{z_j\}$.

Proof of Claim 1: Suppose not. Let x_k be the node of P_{z_1} with the lowest index adjacent to or coincident with a node of $V(P_{z_j}) \cup \{z_j\}$. First

note that x_1 cannot coincide with a node of $V(P_{z_j}) \cup \{z_j\}$, because x_1 is adjacent to a node of $T(\Sigma) \setminus \{h_1, h_3, h_5\}$. If x_k is adjacent to a node of $V(P_{z_j})$ then the path $x_1, \ldots, x_k$ is a direct connection from $T(\Sigma'') \setminus \{h_1'', h_3''\}$ to $B(\Sigma'')$ avoiding $\{h_1'', h_3''\}$ in $G \setminus E(H'')$. This path contradicts Lemma 11.3 because both endnodes of this path are adjacent to a node of $V(\Sigma'') \setminus V(H'')$. Similarily, if node x_k, $k > 1$, is coincident with a node of $V(P_{z_j})$, then the path $x_1, \ldots, x_{k-1}$ contradicts Lemma 11.3 in Σ''. If x_k is adjacent to z_j, then the path $x_1, \ldots, x_k$ contradicts Lemma 11.3 in Σ'' since x_1 is adjacent to a node of $T(\Sigma'') \setminus \{h_1'', h_3'', h_5''\}$ and x_k is not adjacent to any node of $\{h_2, h_4, h_6\}$, so it cannot be a fork of Σ''. If node x_k, $k > 1$, is coincident with y_{m+1} then the path $x_1, \ldots, x_{k-1}$ contradicts Lemma 11.3. This completes the proof of Claim 1.

Claim 2: Node z_1 is not adjacent to or coincident with a node in $V(P_{z_j})$.

Proof of Claim 2: Suppose not. Let y_l, $1 \leq l \leq m$, be the node of the lowest index adjacent to or coincident with the node z_1. If z_1 is adjacent to y_l then the path $x_1, \ldots, x_n, z_1$ is a direct connection from $T(\Sigma'') \setminus \{h_1'', h_3''\}$ to $B(\Sigma'')$ avoiding $\{h_1'', h_3''\}$ in $G \setminus E(H'')$. This path contradicts Lemma 11.3 because both x_1 and z_1 are adjacent to a node of $V(\Sigma'') \setminus V(H'')$. Similarly, if z_1 is a fork of Σ coincident with y_l, then the path $x_1, \ldots, x_n$ contradicts Lemma 11.3. Finally if z_1 is strongly adjacent to Σ with neighbors in both sides of Σ then it cannot coincide with y_l because Σ'' is a connected 6-hole. This completes the proof of Claim 2.

Claim 3: Node z_1 is adjacent to z_j if and only if $j = 2$ or 6.

Proof of Claim 3: Consider the graph G^* induced by the nodes in $T(\Sigma') \cup B(\Sigma'')$.

If $j = 4$ and z_j is adjacent to z_1, then G^* is a connected 6-hole plus the additional edge $z_1 z_4$. If $T(\Sigma')$ is a triad with meet t, then there is a $3PC(z_4, t)$. If $T(\Sigma')$ is a fan $T(\Sigma') \cup \{z_4\}$ induces an odd wheel.

If $j = 2$ or 6, say $j = 2$, and z_j is not adjacent to z_1, then G^* is a connected 6-hole minus the edge $z_1 z_2$. Let P_{13}' be the chordless path between z_1 and h_3 in Σ' and let P_{26}'' be the chordless path between z_2 and h_6 in Σ''. Then there is a $3PC(h_3, h_6)$ unless h_5 has a neighbor in P_{13}' or h_4 has a neighbor in P_{26}''. However in this case there is an odd wheel with center h_5 or h_4. This completes the proof of Claim 3.

So Claims 1, 2 and 3 show $z_j \in H_j(\Sigma')$ completing the proof of the lemma.
□

Corollary 12.6 *Given $z_i \in H_i(\Sigma)$, i even, let Σ_{z_i} be a connected 6-hole obtained from Σ by substituting z_i for h_i. Similarly, given $z_j \in H_j(\Sigma)$, j odd, let Σ_{z_j} be a connected 6-hole obtained from Σ by substituting z_j for h_j. Then z_j can be substituted for h_j in Σ_{z_i} and z_i can be substituted for h_i in Σ_{z_j}.*

Definition 12.7 *Let $T^*(\Sigma)$ be the set of nodes comprising:*

- $T(\Sigma)$
- $\cup_{i \text{ odd}} H_i(\Sigma)$ *together with all the attachments of forks in* $\cup_{i \text{ odd}} H_i(\Sigma)$.

The set $B^*(\Sigma)$ is defined similarily.

An immediate consequence of Lemma 12.5 is the following:

Remark 12.8 *T^* and B^* satisfy the following properties:*

(i) No node of $T^*(\Sigma)$ coincides with a node of $B^*(\Sigma)$.

(ii) Node $w \in T^*(\Sigma)$ is adjacent to node $z \in B^*(\Sigma)$ if and only if $w \in H_i(\Sigma)$ and $z \in H_j(\Sigma)$, for $j = i-1$ or $i+1$. Hence for every node set $\{z_1, \dots, z_6\}$ where $z_i \in H_i(\Sigma)$, $i = 1, \dots, 6$, $z_1, \dots, z_6, z_1$ is a 6-hole.

Property 12.9 *Given nonempty node sets $A_1, \dots, A_6$, that are pairwise disjoint, and node sets Θ_T and Θ_B such that $\cup_{i \text{ odd}} A_i \subseteq \Theta_T$ and $\cup_{i \text{ even}} A_i \subseteq \Theta_B$, we consider a graph $\Theta(\Theta_T, \Theta_B, A_1, \dots, A_6)$ induced by the node set $\Theta_T \cup \Theta_B$ that satisfies the following property:*

(1) Every node u in $\Theta_T \cup \Theta_B$ is contained in some connected 6-hole Σ, such that $T(\Sigma) \subseteq \Theta_T$, $B(\Sigma) \subseteq \Theta_B$ and $h_i \in A_i$ for $i = 1, \dots, 6$. Furthermore if $u \in A_i$, then $u = h_i$.

(2) Let F^t be any triad or fan with attachments $a_i \in A_i$, $i = 1,3,5$ such that $V(F^t) \subseteq \Theta_T$. Let F^b be any triad or fan with attachments $a_i \in A_i$, $i = 2,4,6$ such that $V(F^b) \subseteq \Theta_B$. Then $V(F^t) \cup V(F^b)$ induces a connected 6-hole.

Remark 12.10 *If $\Theta(\Theta_T, \Theta_B, A_1, \dots, A_6)$ satisfies Property 12.9, then it satisfies the following additional properties:*

(1) *Let u be a node in Θ_T and v be a node in Θ_B. Then Θ contains a connected 6-hole Σ such that $u \in T(\Sigma) \subseteq \Theta_T$, $v \in B(\Sigma) \subseteq \Theta_B$ and $h_i \in A_i$, for $i = 1, \ldots, 6$. Furthermore, if $u \in A_i$ for some odd index i, then $u = h_i$. If $v \in A_j$ for some even index j, then $v = h_j$.*

(2) *For every node set $\{a_i \in A_i,\ i = 1, \ldots, 6\}$, $a_1, a_2, a_3, a_4, a_5, a_6, a_1$ is a 6-hole.*

The following procedure constructs a graph, that we will show satisfies Property 12.9.

Initialization: *Set $j = 1$. Let Σ^1 be an arbitrary connected 6-hole of G with 6-hole $H^1 = h_1^1, h_2^1, \ldots, h_6^1, h_1^1$. Let $\Theta_T^1 = T^*(\Sigma^1), \Theta_B^1 = B^*(\Sigma^1)$, $A_i^1 = H_i(\Sigma^1)$ for $i = 1, \ldots, 6$. Let $\Theta^1(\Theta_T^1, \Theta_B^1, A_1^1, \ldots, A_6^1)$ be the graph induced by the node set $\Theta_T^1 \cup \Theta_B^1$. Let $j = 1$ and repeat the following:*

Iterative Step: *If G contains no connected 6-hole Σ satisfying:*

- *$h_i \in A_i^j$ for $i = 1, \ldots, 6$,*
- *Σ is distinct from all Σ^k, $1 \leq k \leq j$, and one of the following two conditions holds:*
  - *(i) $B(\Sigma) = B(\Sigma^k)$ for some $1 \leq k \leq j$, and no node of $T(\Sigma) \setminus \{h_1, h_3, h_5\}$ is adjacent to or coincident with a node of Θ_B^j,*
  - *(ii) $T(\Sigma) = T(\Sigma^k)$ for some $1 \leq k \leq j$, and no node of $B(\Sigma) \setminus \{h_2, h_4, h_6\}$ is adjacent to or coincident with a node of Θ_T^j,*

then stop. Otherwise, let Σ^{j+1} be such a connected 6-hole Σ. Denote the 6-hole of Σ^{j+1} by $H^{j+1} = h_1^{j+1}, h_2^{j+1}, \ldots, h_6^{j+1}, h_1^{j+1}$. Let $\Theta_T^{j+1} = \Theta_T^j \cup T^*(\Sigma^{j+1})$, $\Theta_B^{j+1} = \Theta_B^j \cup B^*(\Sigma^{j+1})$, $A_i^{j+1} = A_i^j \cup H_i(\Sigma^{j+1})$ for $i = 1, \ldots, 6$. Let $\Theta^{j+1}(\Theta_T^{j+1}, \Theta_B^{j+1}, A_1^{j+1}, \ldots, A_6^{j+1})$ be the graph induced by the node set $\Theta_T^{j+1} \cup \Theta_B^{j+1}$. Increment j by 1, and repeat the Iterative Step.

Let w be the index when the above procedure terminates.

To illustrate the procedure, we now apply it to the graph in Figure 7

Let Σ^1 be the connected 6-hole induced by the node set $\{a, \ldots, h, 1, \ldots, 6\}$. No node is attached to Σ^1, so $\Theta_T^1 = \{a, b, c, d, 1, 3, 5\}$, $\Theta_B^1 = \{e, f, g, h, 2, 4, 6\}$, $A_i^1 = \{i\}$ for $i = 1, \ldots, 6$.

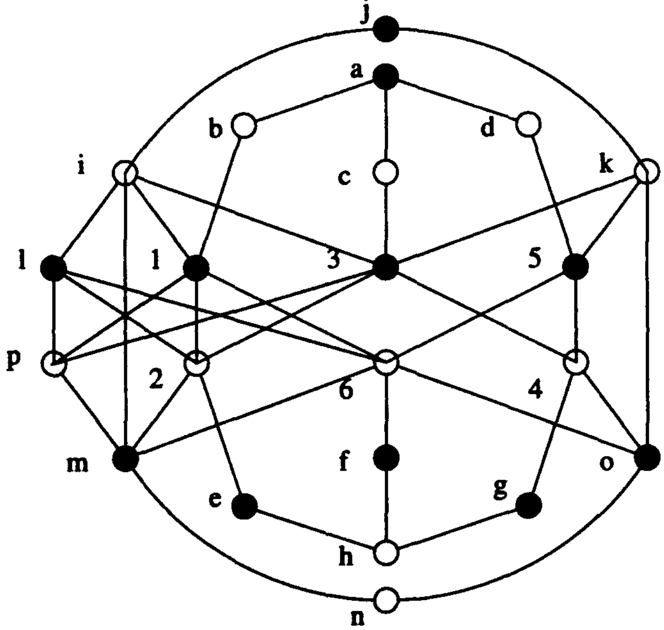


Figure 7: An Example

In the second iteration we can choose Σ^2 such that $B(\Sigma^2) = B(\Sigma^1)$ and $T(\Sigma^2) = \{i, j, k, 1, 3, 5\}$. $\Theta_T^2 = \Theta_T^1 \cup \{i, j, k, l, m, o\}$ and $\Theta_B^2 = \Theta_B^1$. The set A_1^2 now becomes $\{1, l, m\}$, A_5^2 is $\{5, o\}$. $A_i^2 = A_i^1$ for $i = 2, 3, 4, 6$.

The subsequent iterations will enumerate all distinct connected 6-holes with $B(\Sigma^1)$ as bottom and top $\{a_1, 3, a_5, i, j, k\}$ where $a_i \in A_1^2$ and $a_5 \in A_5^2$. The sets Θ_T^2, Θ_B^2, A_i^2, $i = 1, \ldots, 6$ remain unchanged in the subsequent iterations. Note that a different choice of Σ^2, namely one having the same top as Σ^1, would yield different sets A_i^2.

The following lemmas will be used in the proof of the main theorem of this section.

Theorem 12.11 *The graph Θ^w satisfies Property 12.9 .*

Definition 12.12 *Assume that for some $1 \leq n \leq w$, Θ^n satisfies Property 12.9. Then for every $1 \leq i, j \leq n$, the graph induced by the node set $T(\Sigma^i) \cup B(\Sigma^j)$ is a connected 6-hole. We denote this connected 6-hole having top $T(\Sigma^i)$ and bottom $B(\Sigma^j)$ by $\Sigma^{T_i B_j}$.*

Note that the algorithm labels every possible connected 6-hole $\Sigma^{T_i B_j}$ as Σ^k for some $k \leq w$.

Lemma 12.13 *Assume that for some $1 \leq n < w$, Θ^n satisfies Property 12.9. Then $\Theta(\Theta_T^n \cup T(\Sigma^{n+1}), \Theta_B^n \cup B(\Sigma^{n+1}), A_1^n, \ldots, A_6^n)$ satisfies Property 12.9 and for every $k \leq n$, the graphs induced by $T(\Sigma^{n+1}) \cup B(\Sigma^k)$ and $T(\Sigma^k) \cup B(\Sigma^{n+1})$ are the connected 6-holes $\Sigma^{T_{n+1} B_k}$ and $\Sigma^{T_k B_{n+1}}$.*

Proof: The first statement follows from the conditions imposed on Σ^{n+1} by the procedure. The second statement follows from the first and Remark 12.10. □

Lemma 12.14 *Let Σ and Σ' be connected 6-holes such that $T(\Sigma) = T(\Sigma')$ and $h_2 = h_2'$. Let $z_1 \in H_1(\Sigma)$.*

(i) If z_1 is not a fork of Σ, then $z_1 \in H_1(\Sigma')$.

(ii) If z_1 is a fork of Σ, let $P_{z_1} = x_1, \ldots, x_n$ be an attachment of z_1 to Σ. If h_5 is not adjacent to a node of $V(P_{z_1}) \setminus \{x_1\}$, then $z_1 \in H_1(\Sigma')$ and P_{z_1} is an attachment of z_1 in Σ'.

Proof: Let $z_1 \in H_1(\Sigma)$ and, if z_1 is a fork of Σ with attachment $P_{z_1} = x_1, \ldots, x_n$, assume that h_5 is not adjacent to a node of $V(P_{z_1}) \setminus \{x_1\}$. We divide the proof into the following two claims:

Claim 1: No node of P_{z_1} is adjacent to or coincident with a node of $B(\Sigma')$.

Proof of Claim 1: Assume not. Let x_k be the node of P_{z_1} with the lowest index that is adjacent to or coincident with a node of $B(\Sigma')$. First note that x_1 cannot coincide with a node of $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$ because x_1 is adjacent to a node of $T(\Sigma) \setminus \{h_1, h_3, h_5\}$. If node x_k is adjacent to a node of $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$, then $x_1, \ldots, x_k$ is a direct connection from $T(\Sigma') \setminus \{h'_1, h'_3\}$ to $B(\Sigma')$ avoiding $\{h'_1, h'_3\}$ in $G \setminus E(H')$. This path contradicts Lemma 11.3 because both endnodes of this path are adjacent to a node of $V(\Sigma') \setminus V(H')$. Similarily, if node $x_k, k > 1$, is coincident with a node of $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$ then the path $x_1, \ldots, x_{k-1}$ contradicts Lemma 11.3. Node x_k is not adjacent to or coincident with h'_2 since $h'_2 = h_2$ and P_{z_1} is an attachment of z_1 to Σ. Node x_k is not coincident with h'_4 or h'_6, otherwise x_k is adjacent to h_5. Thus node x_k must be adjacent to a node in $\{h'_4, h'_6\}$. But now $x_1, \ldots, x_k$ is a direct connection from $T(\Sigma') \setminus \{h'_1, h'_3\}$ to $B(\Sigma')$ avoiding $\{h'_1, h'_3\}$ in $G \setminus E(H')$, so by Lemma 11.3 node x_k must be a fork of Σ', adjacent to both h'_4 and h'_6. Hence $x_k \in H_5(\Sigma')$. Let Σ'_{x_k} be the connected 6-hole obtained from Σ' by substituting node x_k for the node h_5. If h_1 and h_3 are not adjacent to any node of $x_{k+1}, \ldots, x_n$ then $x_{k+1}, \ldots, x_n, z_1$ is a direct connection between $T(\Sigma'_{x_k})$ and $B(\Sigma'_{x_k}) \setminus \{h'_4, h'_6\}$ avoiding $\{h'_4, h'_6\}$ in $G \setminus \{h'_1h'_2, h'_2h'_3, h'_3h'_4, h'_4x_k, x_kh'_6, h'_6h'_1\}$ and it violates Lemma 11.3. Now assume that h_1 or h_3 is adjacent to some node of $x_k, \ldots, x_n$, and let x_m be the node of $x_k, \ldots, x_n$ with the lowest index adjacent to h_1 or h_3. Assume w.l.o.g. that x_m is adjacent to h_1. Let x_l, $k \leq l < m$ be the node of highest index adjacent to h'_4. Then $x_l, \ldots, x_m$ is a direct connection from $T(\Sigma')$ to $B(\Sigma') \setminus \{h'_6\}$ avoiding $\{h'_6\}$ in $G \setminus E(H')$, violating Lemma 11.2. This completes the proof of Claim 1.

Claim 2: Node z_1 is not adjacent to or coincident with a node of $B(\Sigma') \setminus \{h'_2, h'_6\}$.

Proof of Claim 2: Node z_1 is not adjacent to h'_4, because otherwise the node set $\{h_1, h_2, h_3, h'_4, h_5, h_6\}$ induces an odd wheel with center z_1. If z_1 is strongly adjacent to Σ, with neighbors in $B(\Sigma)$ and $T(\Sigma)$, then z_1 is also strongly adjacent to Σ' with neighbors in $B(\Sigma')$ and $T(\Sigma') = T(\Sigma)$. So by Theorem 10.2 z_1 is adjacent to h'_2, h'_6 but no other node of $B(\Sigma')$. Now

assume z_1 is a fork of Σ. Then z_1 is not coincident with a node of $B(\Sigma')$, else node x_n contradicts Claim 1. Now assume that z_1 is adjacent to a node of $B(\Sigma') \setminus \{h'_2, h'_6, h'_4\}$. Then, by Claim 1, $x_1, \ldots, x_n, z_1$ is a direct connection from $T(\Sigma') \setminus \{h'_1, h'_3\}$ to $B(\Sigma')$ avoiding $\{h'_1, h'_3\}$, in $G \setminus E(H')$ which violates Lemma 11.3. This completes the proof of Claim 2.

Now by Claim 1 and Claim 2, if z_1 is not adjacent to h'_6, then $z_1 \notin H_1(\Sigma')$, so the path $x_1, \ldots, x_n, z_1$ contradicts Lemma 11.3 applied to Σ'. Hence $z_1 \in H_1(\Sigma')$ and if z_1 is a fork of Σ, P_{z_1} is an attachment of z_1 to Σ'. $\square$

Lemma 12.15 *Assume that for some $m \leq w$, the graph Θ^{m-1} satisfies Property 12.9. Let $z_i \in H_i(\Sigma^m)$, i odd, and if z_i is a fork of Σ^m, let P_{z_i} be an attachment of z_i to Σ^m. Then $z_i \in H_i(\Sigma^{T_m B_1})$ and if z_i is a fork of Σ^m, P_{z_i} is an attachment of z_i to $\Sigma^{T_m B_1}$.*

Proof : Assume w.l.o.g. $i = 1$.

Claim 1: Assume that for some $1 < k \leq m$, $z_1 \in H_1(\Sigma^{T_m B_k})$. If z_1 is a fork of $\Sigma^{T_m B_k}$, let P_{z_1} be an attachment of z_1 to $\Sigma^{T_m B_k}$. Then there exists some $j < k$ such that $z_1 \in H_1(\Sigma^{T_m B_j})$ and P_{z_1} is an attachment of z_1 to $\Sigma^{T_m B_j}$.

Proof of Claim 1: By construction $h^k_2 \in A^{k-1}_2$ and $h^k_6 \in A^{k-1}_6$, and hence there exists $i, j < k$ such that $h^k_2 \in H_2(\Sigma^j)$ and $h^k_6 \in H_6(\Sigma^i)$. Let Σ^{jk} be the connected 6-hole obtained from Σ^j by substituting h^k_2 for h^j_2. Let Σ^{ik} be the connected 6-hole obtained from Σ^i by substituting h^k_6 for h^i_6. Since by the definition of attachment, not both h^m_3 and h^m_5 are adjacent to a node of $V(P_{z_1}) \setminus \{x_1\}$, applying Lemma 12.14 either with $\Sigma = \Sigma^{T_m B_k}$ and $\Sigma' = \Sigma^{T_m B_{jk}}$ or with $\Sigma = \Sigma^{T_m B_k}$ and $\Sigma' = \Sigma^{T_m B_{ik}}$ we have that $z_1 \in H_1(\Sigma^{T_m B_{jk}})$ or $z_1 \in H_1(\Sigma^{T_m B_{ik}})$ and P_{z_1} is an attachment of z_1 to one of the two connected 6-holes. Assume w.l.o.g. that $z_1 \in H_1(\Sigma^{T_m B_{jk}})$ and P_{z_1} is an attachment of z_1 to $\Sigma^{T_m B_{jk}}$. Applying again Lemma 12.14 to $\Sigma = \Sigma^{T_m B_{jk}}$ and $\Sigma' = \Sigma^{T_m B_j}$ we have that $z_1 \in H_1(\Sigma^{T_m B_j})$ and P_{z_1} is an attachment of z_1 to $\Sigma^{T_m B_j}$. This completes the proof of Claim 1.

Now the lemma follows by repeated applications of Claim 1, starting with $\Sigma^m = \Sigma^{T_m B_m}$. $\square$

Lemma 12.16 *Let Σ and Σ' be connected 6-holes such that $T(\Sigma) = T(\Sigma')$ and $h_2 = h'_2$. Let $z_1 \in H_1(\Sigma)$.*

(i) If z_1 is not a fork of Σ then $z_1 \in H_1(\Sigma')$.

(ii) *If z_1 is a fork of Σ, let P_{z_1} be an attachment of z_1 to Σ. If no node of P_{z_1} is adjacent to or coincident with h'_4 or h'_6 then $z_1 \in H_1(\Sigma')$ and P_{z_1} is an attachment of z_1 to Σ'.*

Proof: Let $z_1 \in H_1(\Sigma)$ and if z_1 is a fork of Σ then let $P_{z_1} = x_1, \ldots, x_n$ be an attachment of z_1 to Σ. Suppose that P_{z_1} satisfies the conditions of the lemma. Assume w.l.o.g. that h_5 is not adjacent to a node of $V(P_{z_1}) \setminus \{x_1\}$.

Claim 1: No node of P_{z_1} is adjacent to or coincident with a node of $B(\Sigma')$.

Proof of Claim 1: Assume not. Let x_k be the node of P_{z_1} with the lowest index that is adjacent to or coincident with a node of $B(\Sigma')$. First note that x_1 cannot coincide with a node of $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$ because x_1 is adjacent to a node of $T(\Sigma) \setminus \{h_1, h_3, h_5\}$. If node x_k is adjacent to a node of $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$, then $x_1, \ldots, x_k$ is a direct connection from $T(\Sigma') \setminus \{h'_1, h'_3\}$ to $B(\Sigma')$ avoiding $\{h'_1, h'_3\}$ in $G \setminus E(H')$. This path contradicts Lemma 11.3 because both endnodes of this path are adjacent to a node of $V(\Sigma') \setminus V(H')$. Similarily, if node $x_k, k > 1$, is coincident with a node of $B(\Sigma') \setminus \{h'_2, h'_4, h'_6\}$ then the path $x_1, \ldots, x_{k-1}$ contradicts Lemma 11.3. Node x_k is not adjacent to or coincident with h'_2 since $h'_2 = h_2$ and P_{z_1} is an attachment of z_1 to Σ. This completes the proof of Claim 1.

Claim 2: Node z_1 is not adjacent to or coincident with a node of $B(\Sigma') \setminus \{h'_2, h'_6\}$.

Proof of Claim 2: Identical to proof of Claim 2 in Lemma 12.14.

Now by Claim 1 and Claim 2, if z_1 is not adjacent to h'_6, then $z_1 \notin H_1(\Sigma')$, so the path $x_1, \ldots, x_n, z_1$ contradicts Lemma 11.3 applied to Σ'. Hence $z_1 \in H_1(\Sigma')$. □

Proof of Theorem 12.11: Let n be the smallest index for which Θ^n does not satisfy Property 12.9. Lemma 12.5 and Corollary 12.6 show that Θ^1 satisfies Property 12.9. Hence $n > 1$. Furthermore Lemma 12.13 shows that $\Theta(\Theta_T^{n-1} \cup T(\Sigma^n), \Theta_B^{n-1} \cup B(\Sigma^n), A_i^{n-1},\ i = 1 \ldots, 6)$ satisfies Property 12.9. By construction of Θ^n and by Lemma 12.5 applied to every $\Sigma^{T_n B_k}$, $k \leq n$, Property 12.9 holds for Θ^n if the following holds:

Let $z_i \in H_i(\Sigma^n)$, i odd, and if z_i is a fork of Σ^n, let P_{z_i} be an attachment of z_i to Σ^n. Let $z_j \in H_j(\Sigma^k)$, where j is even and $k \leq n$. If z_j is a fork of Σ^k, let P_{z_j} be an attachment of z_j to Σ^k. Then $z_i \in H_i(\Sigma^{T^n B^k})$ and P_{z_i} is an

attachment of z_i to $\Sigma^{T_n B_k}$ and $z_j \in H_j(\Sigma^{T^n B^k})$ and P_{z_j} is an attachment of z_j to $\Sigma^{T_n B_k}$.

The next two claims prove the above statement.

Claim 1: Let $z_j \in H_j(\Sigma^k)$,where j is even and $k \leq n$. If z_j is a fork of Σ^k, let P_{z_j} be an attachment of z_j to Σ^k. Then $z_j \in H_j(\Sigma^{T^n B^k})$ and P_{z_j} is an attachment of z_j to $\Sigma^{T_n B_k}$.

Proof of Claim 1: If $k = n$ the claim follows from Corollary 4.6. If $k < n$, then z_j together with $V(P_{z_j})$ belongs to Θ_B^{n-1}. hence the claim follows by construction of Σ^n. This completes the proof of Claim 1.

Claim 2: Let $z_i \in H_i(\Sigma^n)$, i odd, and if z_i is a fork of Σ^n, let P_{z_i} be an attachment of z_i to Σ^n. Then for every $k \leq n$, $z_i \in H_i(\Sigma^{T^n B^k})$ and P_{z_i} is an attachment of z_i to $\Sigma^{T_n B_k}$.

Proof of Claim 2: Assume w.l.o.g. that $i = 1$. Let $k < n$ be the smallest index for which $z_1 \notin H_1(\Sigma^{T_n B_k})$. Then by Lemma 12.15, $k > 1$. By construction, h_2^k in A_2^{k-1}. Hence there exists a $j \leq k-1$ such that $h_2^k \in H_2(\Sigma^j)$. Then, by the choice of k, $z_1 \in H_1(\Sigma^{T_n B_j})$ and by Claim 1, $h_2^k \in H_2(\Sigma^{T_n B_j})$. Now Lemma 12.5 applied to $\Sigma^{T_n B_j}$ shows that z_1 is adjacent to h_2^k and no node of P_{z_1} is adjacent to or coincident with h_2^k. The same argument shows that z_1 is adjacent to h_6^k but not to h_4^k and no node of P_{z_1} is adjacent to or coincident with h_4^k of h_6^k. Let Σ^{jk} be the connected 6-hole obtained by substituting h_2^k for h_2^j in Σ^j. Consider the connected 6-hole $\Sigma^{T_n B_{jk}}$. Now by Lemma 12.16 applied to $\Sigma = \Sigma^{T_n B_j}$ and $\Sigma' = \Sigma^{T_n B_{jk}}$, we have that $z_1 \in H_1(\Sigma^{T_n B_{jk}})$ and P_{z_1} is an attachment of z_1 in $\Sigma^{T_n B_{jk}}$. Applying again Lemma 12.16 to $\Sigma = \Sigma^{T_n B_{jk}}$ and $\Sigma' = \Sigma^{T_n B_k}$, we have that $z_1 \in H_1(\Sigma^{T_n B_k})$ and P_{z_1} is an attachment of z_1 in $\Sigma^{T_n B_k}$. This completes the proof of Claim 2. □

Corollary 12.17 *If Σ is a connected 6-hole such that $T(\Sigma) \subseteq \Theta_T^w$, $B(\Sigma) \subseteq \Theta_B^w$, and $h_i \in A_i^w$ for $i = 1, \ldots, 6$, then Σ coincides with Σ^k, for some $1 \leq k \leq w$.*

Proof: Suppose that Σ does not coincide with any Σ^i, $i = 1, \ldots, w$. First we show that for some $1 \leq k \leq w$, $T(\Sigma) = T(\Sigma^k)$. Suppose not. Then since by Theorem 12.11 Θ^w satisfies Property 12.9 (2), Σ' such that $T(\Sigma') = T(\Sigma)$ and $B(\Sigma') = B(\Sigma^1)$ is a connected 6-hole satisfying the rules of construction. Hence for some $1 \leq k \leq w$, Σ' coincides with Σ^k. Now Σ is such that $T(\Sigma) = T(\Sigma^k)$ and no node of $B(\Sigma) \setminus \{h_2, h_4, h_6\}$ is adjacent to or coincident with a node of Θ_T^w. Hence it is possible to define Σ^{w+1}, which contradicts the maximality of w. □

Corollary 12.18 *If $u \in \Theta_T^w$ and $v \in \Theta_B^w$, then for some $1 \leq k \leq w$, nodes u and v are contained in Σ^k.*

Proof: Follows from Theorem 12.11, Remark 12.10 (1) and Corollary 12.17. □

13 Extended Star Cutset

Definition 13.1 *Let A be the graph induced by the node set $\cup_{i=1}^{6} A_i^w$.*

Lemma 13.2 *If the removal of the edge set $E(A)$ disconnects the graph G, then G contains a 6-join.*

Proof: Assume that the edge set $E(A)$ disconnects the graph G. By Theorem 12.11, Θ^w satisfies Property 12.9. Now by Remark 12.10 (2), $E(A)$ is a 6-join of G. □

In this section we prove the following theorem.

Theorem 13.3 *G contains a 6-join or an extended star cutset.*

Lemma 13.4 *If $P = x_1, \ldots, x_n$ is a direct connection from $\Theta_T^w \setminus (A_1^w \cup A_3^w)$ to Θ_B^w avoiding $A_1^w \cup A_3^w$ in $G \setminus E(A)$, then $N(x_1) \cap V(\Theta^w) \subseteq A_1^w \cup A_3^w \cup A_5^w$, $N(x_1) \cap A_5^w \neq \emptyset$ and $N(x_n) \cap V(\Theta^w) \subseteq A_2^w \cup A_4^w \cup A_6^w$.*

Proof: If x_1 is adjacent to a node $v \in \Theta_B^w$, then $n = 1$, and let u be any neighbor of x_1 in Θ_T^w. By Corollary 12.18, for some $1 \leq k \leq w$, nodes u and v are contained in Σ^k. Node x_1 is now strongly adjacent to Σ^k, with neighbors in both sides of Σ^k. Hence for some $i \in \{1, \ldots, 6\}$, $x_1 \in H_i(\Sigma^k)$. By construction of Θ^w, $x_1 \in V(\Theta^w)$ which contradicts our choice of P. Therefore $N(x_1) \cap V(\Theta^w) \subseteq \Theta_T^w$. Now suppose that x_1 is adjacent to a node $u \in \Theta_T^w \setminus (A_1^w \cup A_3^w \cup A_5^w)$. Let v be any neighbor of x_n in Θ_B^w. By Corollary 12.18, for some $1 \leq k \leq w$, nodes u and v are contained in Σ^k. Now by Lemma 11.3 and Definition 12.1, x_n is an attached node with respect to Σ^k. By construction of Θ^w, $x_n \in \Theta_T^w$ which contradicts our choice of P. Therefore $N(x_1) \cap V(\Theta^w) \subseteq A_1^w \cup A_3^w \cup A_5^w$. Similarily $N(x_n) \cap V(\Theta^w) \subseteq A_2^w \cup A_4^w \cup A_6^w$. □

Definition 13.5 *A bridge of Type c with respect to a connected 6-hole Σ, is a configuration C satisfying the following properties:*

- *C is connected.*
- *There exist nodes $h_{i-1}, h_i, h_{i+1}, h_{i+2}$ of Σ that are adjacent to at least one node of C. No other node of Σ is adjacent to a node of C.*
- *C is minimal with the above two properties.*

Note that bridges of Type c1 and Type c2 defined in Lemma 11.1 and Lemma 11.2 are bridges of Type c.

Lemma 13.6 *Let C be a bridge of Type c. Then C satisfies the following property:*

- *C induces a path $P = x_1, \ldots, x_n$ which is a direct connection between h_{i-1} and h_{i+2} in C.*
- *P contains at least one node adjacent to h_i and at least one node adjacent to h_{i+1}.*

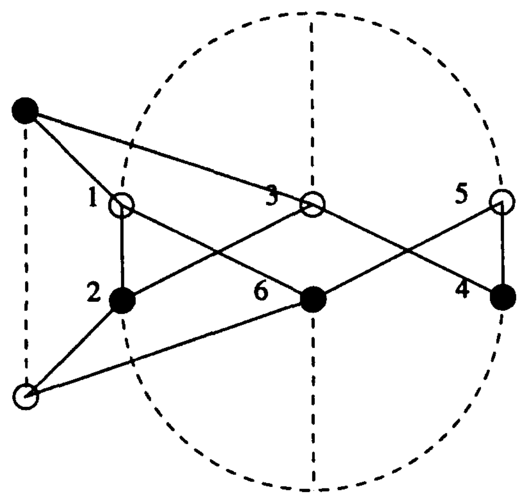
Proof: Assume w.l.o.g. that $i = 1$. Let P be a direct connection in C between h_3 and h_6. If P satisfies the second property of the lemma, then by minimality of C, P and C coincide. If P has no node adjacent to h_2, then P is a direct connection between $T(\Sigma) \setminus \{h_1\}$ and $B(\Sigma)$ avoiding $\{h_1\}$ in $G \setminus E(H)$ violating Lemma 11.2. If P has no node adjacent to h_1, the proof is identical. □

Figure 8 depicts possible bridges of Type c.

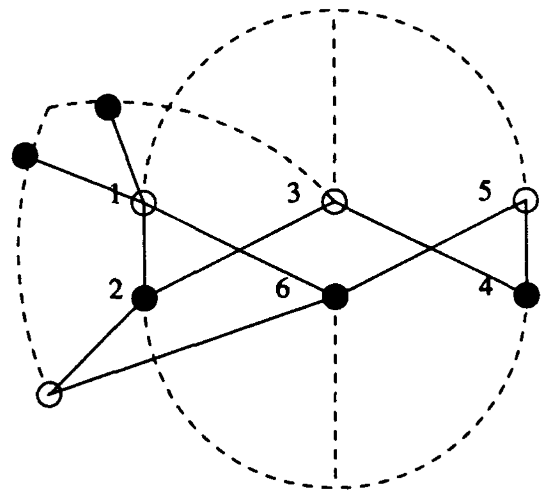
Lemma 13.7 *Let P be a bridge of Type c with respect to Σ^k, for some $1 \le k \le w$, containing nodes adjacent to $h_1^k, h_2^k, h_3^k, h_6^k$. If no node of P is adjacent to any node in $V(\Theta^w) \setminus V(A)$, then every node in $A_1^w \cup A_2^w$ has a neighbor on P.*

Proof: Let Σ^k and P satisfy the conditions of the lemma. It suffices to prove the result for A_2^w.

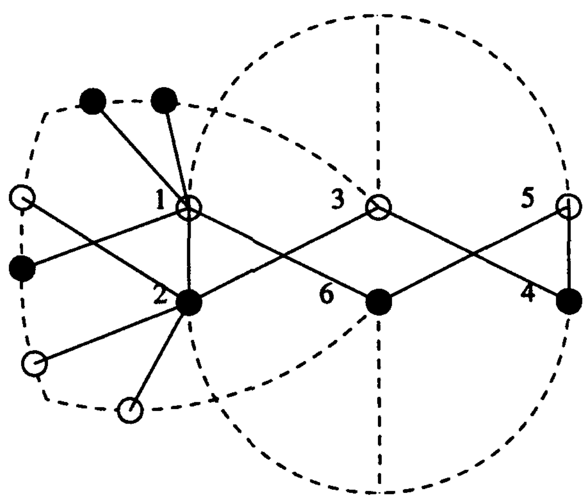
Claim 1: For $1 \le m \le w$, if some node $x \in H_2(\Sigma^m)$ has a neighbor on P, then every node in $H_2(\Sigma^m)$ has a neighbor on P.



Bridge of Type c1



Bridge of Type c2



Bridge of Type c

Figure 8: Bridges of Type c

Proof of Claim 1: Let $y \in H_2(\Sigma^m)$, and suppose that y is not adjacent to any node of P. Let Σ^m_x (resp. Σ^m_y) be a connected 6-hole obtained from Σ^m by substituting node x (resp. y) for h^m_2.

If there exists a chordless path Q from h^k_6 to y using only nodes in $(\Theta^w_B \setminus (A^w_2 \cup A^w_4 \cup A^w_6)) \cup \{h^k_6, y\}$, then there exists a $3PC(h^k_3, h^k_6)$ using P, Q and a path connecting h^k_3 and h^k_5 in $T(\Sigma^k)$. So assume that no such path exists. In particular h^k_6 is not adjacent to any node in $B(\Sigma^m_x) \cup B(\Sigma^m_y)$. Now note that $V(P) \cup \{h^k_6, h^k_1, h^k_3, h^k_5\}$ induces a fan top attached to h^k_1, h^k_3, h^k_5, with center h^k_1, which is node disjoint from $B(\Sigma^m_y)$ and has no adjacencies to $B(\Sigma^m_y) \setminus \{y, h^m_4, h^m_6\}$. Let Σ be the connected 6-hole induced by the node set $V(P) \cup \{h^k_6, h^k_1, h^k_3, h^k_5\} \cup B(\Sigma^m_y)$. The 6-hole of Σ is $h^k_1, y, h^k_3, h^m_4, h^k_5, h^m_6, h^k_1$. Since both $x, y \in H_2(\Sigma^m)$, by Lemma 12.5 $x \in H_2(\Sigma^m_y)$. If x has neighbors in both sides of Σ^m_y then x is adjacent to a node in $B(\Sigma^m_y) \setminus \{y, h^m_6, h^m_4\}$. By assumption x is adjacent to P. Hence x is a strongly adjacent node to Σ which violates Theorem 10.2. If x is a fork of Σ^m_y, let $x_1, \ldots, x_n$ be an attachment of x to Σ^m_y. We can assume w.l.o.g. that $x_1, \ldots, x_n, x$ is a direct connection from $B(\Sigma^m_y) \setminus \{y, h^m_6\}$ to $T(\Sigma^m_y)$ avoiding $\{y, h^m_6\}$ in $G \setminus \{h^m_1 y, y h^m_3, h^m_3 h^m_4, h^m_4 h^m_5, h^m_5 h^m_6, h^m_6 h^m_1\}$. Hence x_1 is adjacent to $B(\Sigma^m_y) \setminus \{y, h^m_4, h^m_6\}$. By assumption x has a neighbor on P, hence $x_1, \ldots, x_n, x$ is a direct connection from $B(\Sigma) \setminus \{y, h^m_6\}$ to $T(\Sigma)$ avoiding $\{y, h^m_6\}$ in $G \setminus \{h^k_1 y, y h^k_3, h^k_3 h^m_4, h^m_4 h^k_5, h^k_5 h^m_6, h^m_6 h^k_1\}$, violating Lemma 11.3. This completes the proof of Claim 1.

Claim 2: If for some $1 < n \leq w$, every node in $H_2(\Sigma^n)$ has a neighbor on P, then there exists $1 \leq m < n$ such that every node in $H_2(\Sigma^m)$ has a neighbor on P.

Proof of Claim 2: Assume that $1 < n \leq w$, and every node in $H_2(\Sigma^n)$ has a neighbor on P. In particular h^n_2 is adjacent to a node of P. But by construction $h^n_2 \in A^{n-1}_2$, thus there exists $1 \leq m < n$ such that $h^n_2 \in H_2(\Sigma^m)$. Now by Claim 1, every node in $H_2(\Sigma^m)$ has a neighbor on P. This completes the proof of Claim 2.

Now we show that for every $1 \leq n \leq w$, every node in A^n_2 has a neighbor on P, by induction on n. By Claim 1 every node in $H_2(\Sigma^k)$ has a neighbor on P, so by repeated application of Claim 2, every node in $H_2(\Sigma^1) = A^1_2$ has a neighbor on P, hence the base case holds. Now assume that for $1 \leq n < w$, every node in A^n_2 has a neighbor on P. By construction $h^{n+1}_2 \in A^n_2$, hence by Claim 1 every node in $H_2(\Sigma^{n+1})$ has a neighbor on P. Thus every node in A^{n+1}_2 has a neighbor on P. This completes the proof of the lemma. □

Lemma 13.8 *Let $P = x_1, \ldots, x_n$ be a direct connection from $\Theta_T^w \setminus (A_1^w \cup A_5^w)$ to $\Theta_B^w \setminus (A_2^w \cup A_4^w)$ avoiding $A_1^w \cup A_5^w \cup A_2^w \cup A_4^w$ in $G \setminus E(A)$. If for some $1 \leq k \leq w$, Σ^k is such that x_1 is adjacent to a node of $T(\Sigma^k) \setminus \{h_1^k, h_5^k\}$ and x_n is adjacent to a node of $B(\Sigma^k) \setminus \{h_2^k, h_4^k\}$, then P is a bridge of Type c with respect to Σ^k, with nodes $h_1^k, h_2^k, h_3^k, h_6^k$ adjacent to at least one node in P or nodes $h_3^k, h_4^k, h_5^k, h_6^k$ adjacent to P.*

Proof: Let $\Sigma = \Sigma^k$ and P satisfy the conditions of the lemma. Let x_i be the node of P with the lowest index that is adjacent to a node in Θ_B^w. Then $P_{x_1 x_i}$ is a direct connection from $\Theta_T^w \setminus (A_1^w \cup A_5^w)$ to Θ_B^w avoiding $A_1^w \cup A_5^w$ in $G \setminus E(A)$, so by Lemma 13.4 $N(x_1) \cap V(\Theta^w) \subseteq A_1^w \cup A_3^w \cup A_5^w$. In particular x_1 is adjacent to h_3. Similarily $N(x_n) \cap V(\Theta^w) \subseteq A_2^w \cup A_4^w \cup A_6^w$ and x_n is adjacent to h_6. Let x_j be the node of lowest index adjacent to $B(\Sigma)$. Then the path $P_{x_1 x_j}$ is a direct connection from $T(\Sigma) \setminus \{h_1, h_5\}$ to $B(\Sigma)$ avoiding $\{h_1, h_5\}$ in $G \setminus E(H)$, so by Lemma 11.3, node x_j is adjacent to some node in $\{h_2, h_4\}$. Similarily some node in $\{h_1, h_5\}$ is adjacent to a node of P.

In the following claim we prove the lemma, with the restriction that no node in one of the sets A_1^w, A_2^w, A_4^w or A_5^w has a neighbor on P.

Claim: If no node in one of the sets A_1^w, A_2^w, A_4^w or A_5^w has a neighbor on P, then P is a bridge of Type c with respect to Σ.

Proof of Claim: Assume w.l.o.g. that no node in A_5^w has a neighbor in $V(P)$. First suppose that $V(P)$ does not contain neighbors of both h_2 and h_4. If h_4 is adjacent to a node of P, then some subpath of P is a direct connection from $B(\Sigma)$ to $T(\Sigma) \setminus \{h_1\}$ avoiding $\{h_1\}$ in $G \setminus E(H)$ which contradicts Lemma 11.2. Thus P must contain nodes adjacent to h_1 and h_2, and no node adjacent to h_4 and h_5. Hence P is a bridge of Type c with respect to Σ.

We now show that P cannot have neighbors from both h_2 and h_4. Assume the contrary. Let x_j be the node of highest index in P adjacent to a node in $A_1^w \cup A_3^w$. Let x_j be adjacent to node h' in $A_1^w \cup A_3^w$. By Corollary 12.18, let Σ' be a connected 6-hole of Θ^w containing nodes h' and $h'_6 = h_6$. By Corollary 12.17 we can assume that $B(\Sigma') = B(\Sigma)$. Node h'_4 must have a neighbor on P since $h_4 = h'_4$. If $P_{x_1 x_j}$ has a neighbor of h'_4, then $h' = h'_1$, since h'_3 can only be adjacent to node x_1 on P. Let $x_p \in V(P_{x_1 x_j})$ be the node of highest index adjacent to h'_4. Now some subpath of $P_{x_p x_j}$ is a direct connection between $T(\Sigma')$ and $B(\Sigma') \setminus \{h'_2\}$ avoiding $\{h'_2\}$ in $G \setminus E(H')$ which violates Lemma 11.2. So h'_4 must have a neighbor on $P_{x_j x_n}$. Let $x_p \in V(P_{x_j x_n})$ be the node of

lowest index adjacent to h_4'. Now $P_{x_px_j}$ is a direct connection between $T(\Sigma')$ and $B(\Sigma') \setminus \{h_2'\}$ avoiding $\{h_2'\}$ in $G \setminus E(H')$. By Lemma 11.2, x_j must be a fork of Σ' adjacent to h_1' and h_3'. But h_3' can only be adjacent to x_1 on P. So $j = 1$ and no node of A_1^w is adjacent to a node of $P_{x_2x_n}$. Now let x_k be the node of highest index in P adjacent to a node in $\{h_2, h_4\}$. If x_k is adjacent to h_4 then let x_l be the node of highest index adjacent to h_2. The subpath $P_{x_lx_n}$ has no neighbors of $A_1^w \cup A_5^w$, thus the node set $\{h_2, h_4, h_6, x_l, \ldots, x_n\}$ induces a fan bottom with center h_4, which contradicts the maximality of w. Similarily if x_k is adjacent to h_2 then we can obtain a fan bottom with center h_2. This completes the proof of Claim.

If one of the node sets $A_1^w, A_2^w, A_4^w, A_5^w$ has no node adjacent to P then we are done by the Claim above. So assume all four node sets have at least one node adjacent to a node in $V(P)$. Let x_k be the node of highest index adjacent to a node in $A_1^w \cup A_5^w$. Assume w.l.o.g. it is adjacent to $h_1' \in A_1^w$. Notice that $k > 1$ since otherwise x_1 is adjacent to $a_1 \in A_1^w$, $a_3 \in A_3^w$ and $a_5 \in A_5^w$ and so we have an odd wheel with center x_1.

First we show that no node of A_4^w is adjacent to $P_{x_1x_k}$. Assume not and let x_l be the node of $P_{x_1x_k}$ with the highest index adjacent to a node in A_4^w. Let h_4'' be the node of A_4^w adjacent to x_l. Let x_m be the node of $P_{x_lx_k}$ with the lowest index adjacent to a node in A_1^w, and let h_1'' be that node. By Corollary 12.18 let Σ'' be a connected 6-hole of Θ^w containing nodes h_4'' and h_1''. Now $P_{x_lx_m}$ is a direct connection from $\Theta_B^w \setminus (A_2^w \cup A_6^w)$ to $\Theta_T^w \setminus (A_3^w \cup A_5^w)$ avoiding $A_2^w \cup A_3^w \cup A_5^w \cup A_6^w$ in $G \setminus E(A)$. Also Σ'' is such that x_m is adjacent to h_1'', x_l is adjacent to h_4'' and no node of A_3^w is adjacent to a node in $V(P_{x_lx_m})$ (since the only node of P that can have a neighbor in A_3^w is x_1). Now by the Claim, $P_{x_lx_m}$ is a bridge of Type c with respect to Σ'', with neighbors from h_5'' and h_6''. But the only neighbor h_6'' can have on P is x_n, hence we have a contradiction. Therefore no node of A_4^w is adjacent to $P_{x_1x_k}$.

Let x_l be the node of $P_{x_kx_n}$ with the lowest index adjacent to some node A_4^w, and let h_4' be that node. By Corollary 12.18 let Σ' be a connected 6-hole of Θ^w containing nodes h_1' and h_4'. $P_{x_kx_l}$ is a direct connection from $\Theta_T^w \setminus (A_3^w \cup A_5^w)$ to $\Theta_B^w \setminus (A_2^w \cup A_6^w)$ avoiding $A_2^w \cup A_3^w \cup A_5^w \cup A_6^w$ in $G \setminus E(A)$. Also Σ' is such that x_k is adjacent to h_1', x_l is adjacent to h_4' and no node of A_3^w is adjacent to a node in $V(P_{x_kx_l})$ (since the only node of P that can have a neighbor in A_3^w is x_1 and $k > 1$). Now by the Claim, $P_{x_kx_l}$ must be a bridge of Type c with respect to Σ', with neighbors from h_5' and h_6' and no neighbor of h_2' and h_3'. By Lemma 13.7 every node in A_5^w is adjacent to

a node in $P_{x_k x_l}$. By our choice of x_k all nodes in A_5^w must be adjacent to x_k. Also since h_6' is adjacent to $P_{x_k x_l}$ we must have $l = n$. If any node in A_2^w is adjacent to a node of $P_{x_k x_n}$ then let x_p be the node of $P_{x_k x_n}$ of lowest index adjacent to a node in A_2^w, say h_2''. Note that $p < n$. By Corollary 12.18 let Σ'' be a connected 6-hole of Θ^w containing h_2'' and $h_1'' = h_1'$. Now $P_{x_k x_p}$ is a direct connection from $T(\Sigma'')$ to $B(\Sigma'')$ in $G \setminus E(H'')$ which contradicts Lemma 11.1 since $p < n$ and so h_4' and h_6' are not adjacent to x_p. Thus no node of A_2^w is adjacent to a node of $P_{x_k x_n}$.

Now since some node of A_2^w must be adjacent to a node of P, this node ıst be in $P_{x_1 x_k}$. Let x_p be the node of $P_{x_1 x_k}$ with the highest index adjacent ﹚ some node in A_2^w and let that node be h_2''. Let x_q be the node of lowest index in $P_{x_p x_k}$ adjacent to a node in A_5^w and let that node be h_5''. Notice that such a node must exist since every node in A_5^w is adjacent to x_k. By Corollary 12.18, let Σ'' be a connected 6-hole of Θ^w containing h_5'' and h_2''. Now $P_{x_p x_q}$ is a direct connection from $\Theta_B^w \setminus (A_4^w \cup A_6^w)$ to $\Theta_T^w \setminus (A_1^w \cup A_3^w)$ avoiding $A_1^w \cup A_3^w \cup A_4^w \cup A_6^w$ in $G \setminus E(A)$. Also Σ'' is such that x_p is adjacent to h_2'', x_q is adjacent to h_5'' and no node of A_3^w is adjacent to a node of $P_{x_p x_q}$ (since the only neighbor that a node in A_3^w can have on P is x_1). But now by the Claim, $P_{x_p x_q}$ is a bridge of Type c with respect to Σ'', with neighbors from h_1'' and h_6''. But the only neighbor that h_6'' can have on P is x_n, hence we have a contradiction. But then A_2^w does not have any node adjacent to a node of P, which contradicts our assumption. □

Lemma 13.9 *Let $P = x_1, \ldots, x_n$ be a bridge of Type c with respect to Σ^k, for $1 \le k \le w$, with adjacencies to h_1^k, h_2^k, h_3^k and h_6^k, where x_1 is adjacent to h_3^k. Then $A_1^w \cup A_2^w \cup A_3^w \cup N(h_2^k)$ is an extended star cutset seperating x_1 from Θ^w.*

Proof: Let P and Σ satisfy the conditions of the lemma. Let $R = A_1^w \cup A_2^w \cup A_3^w \cup N(h_2)$ and suppose that R is not an extended star cutset. Let $Q = y_1, \ldots, y_m$ be a direct connection from x_1 to $\Theta^w \setminus R$ in $G \setminus R$. By Lemma 13.4, y_m cannot have neighbors in both Θ_T^w and Θ_B^w. Let $Q' = y_0, y_1, \ldots, y_m$ where $y_0 = x_1$.

Case 1: $N(y_m) \cap V(\Theta^w) \subseteq \Theta_B^w$

Some subpath of Q' is a direct connection from Θ_T^w to $\Theta_B^w \setminus (A_2^w \cup A_4^w)$ avoiding $A_2^w \cup A_4^w$ in $G \setminus E(A)$ or a direct connection from Θ_T^w to $\Theta_B^w \setminus (A_2^w \cup A_6^w)$ avoiding $A_2^w \cup A_6^w$ in $G \setminus E(A)$. In either case, by Lemma 13.4, $N(y_m) \cap V(\Theta^w) \subseteq A_2^w \cup A_4^w \cup A_6^w$. Hence y_m is adjacent to a node in $A_4^w \cup A_6^w$.

Suppose that y_m is adjacent to a node $x \in A_6^w$. Let y_i be the node of Q' with highest index adjacent to a node in A_3^w. Then $Q'_{y_i y_m}$ is a direct connection from $\Theta_T^w \setminus (A_1^w \cup A_5^w)$ to $\Theta_B^w \setminus (A_2^w \cup A_4^w)$ avoiding $A_1^w \cup A_5^w \cup A_2^w \cup A_4^w$ in $G \setminus E(A)$. By Corollary 12.18, let Σ' be a connected 6-hole of Θ^w containing node x and a node of A_3^w that is adjacent to y_i. By Lemma 13.8 applied to $Q'_{y_i y_m}$ and Σ', $Q'_{y_i y_m}$ is a bridge of Type c with respect to Σ'. Since $Q'_{y_i y_m}$ is not adjacent to any node in A_5^w, $Q'_{y_i y_m}$ is adjacent to h'_1 and h'_2. Now by Lemma 13.7 every node in A_2^w has a neighbor on $Q'_{y_i y_m}$. In particular h_2 is adjacent to Q, contradicting our choice of Q. Therefore y_m is not adjacent to any node in A_6^w.

Now suppose that y_m is adjacent to a node $x \in A_4^w$. First we will show that no node of A_1^w is adjacent to a node of Q'. Assume not and let y_i be the node of Q' with the highest index adjacent to a node in A_1^w. Then Q'_{y_i, y_m} is a direct connection from $\Theta_T^w \setminus (A_3^w \cup A_5^w)$ to $\Theta_B^w \setminus (A_2^w \cup A_6^w)$ avoiding $(A_3^w \cup A_5^w \cup A_2^w \cup A_6^w)$ in $G \setminus E(A)$. By Corollary 12.18, let Σ' be a connected 6-hole of Θ^w containing node x and a node of A_1^w adjacent to y_i. Then by Lemma 13.8, $Q'_{y_i y_m}$ is a bridge of Type c with respect to Σ'. Since Q is not adjacent to any node in A_5^w, h'_2 is adjacent to $Q'_{y_i y_m}$, and by Lemma 13.7 every node of A_2^w has a neighbor on Q. In particular h_2 is adjacent to Q, which contradicts our choice of Q. Therefore no node of A_1^w is adjacent to a node of Q'.

Now let x_j be the node of P with the lowest index adjacent to a node of A_1^w. Let y_i be the node of Q' of highest index adjacent to a node of $P_{x_1 x_j}$. Let x_l be the node of $P_{x_1 x_j}$ with highest index adjacent to y_i. By the same argument as above, the path induced by the node set $V(P_{x_l x_j}) \cup V(Q'_{y_i y_m})$ must have a neighbor of h_2 and a neighbor of h_3 on it. By construction of Q the neighbor of h_2 is on $P_{x_l x_j}$. Let x_s be the neighbor of h_2 on $P_{x_l x_j}$ with the lowest index. By construction of P, h_3 has no neighbors on $P_{x_s x_j}$. By Corollary 12.17, let Σ' be a connected 6-hole of Θ^w with $h'_4 = x$. Let Y be the path connecting h'_6 and x in Σ'. If there exists a chordless path X from x to h_2 using nodes in Θ_B^w only then there are two wheels with center h_3: $x_s, \ldots, x_l, y_i, \ldots, y_m, X, x_s$ and $x_j, \ldots, x_l, y_i, \ldots, y_m, Y, h_1, x_j$. One of these wheels must be odd, thus we have a contradiction. Otherwise, if no such path X exists, x has no neighbors in $B(\Sigma)$. Now the path $x_s, \ldots, x_l, y_i, \ldots, y_m, x$ is a direct connection from $B(\Sigma)$ to $T(\Sigma) \setminus \{h_1, h_3\}$ avoiding $\{h_1, h_3\}$ in $G \setminus E(H)$. Since x is adjacent to h_3 and h_5, by Lemma 11.3 x_s is adjacent to h_4, which contradicts our choice of P.

Case 2: $N(y_m) \cap V(\Theta^w) \subseteq \Theta_T^w$

If some node of A_2^w has a neighbor on Q, let y_i be the node of highest index adjacent to some node in A_2^w, and let $Y = Q_{y_i y_m}$. Otherwise let x_i be the node of P with the lowest index adjacent to some node in A_2^w, and let Y be the path induced by the node set $V(Q) \cup V(P_{x_1 x_i})$. Then Y is a direct connection from $\Theta_T^w \setminus (A_1^w \cup A_3^w)$ to Θ_B^w avoiding $A_1^w \cup A_3^w$ in $G \setminus E(A)$. By Corollary 13.4 $N(y_m) \cap V(\Theta^w) \subseteq A_1^w \cup A_3^w \cup A_5^w$. Hence y_m is adjacent to some node $x \in A_5^w$.

Let x_j be the node of P with lowest index adjacent to a node of A_2^w. Let y be a node of A_2^w adjacent to x_j. Path X induced by the node set $V(P_{x_1 x_j}) \cup V(Q)$ is a direct connection from $\Theta_T^w \setminus (A_1^w \cup A_3^w)$ to $\Theta_B^w \setminus (A_4^w \cup A_6^w)$ avoiding $A_1^w \cup A_3^w \cup A_4^w \cup A_6^w$ in $G \setminus E(A)$. By Corollary 12.18, let Σ' be a connected 6-hole of Θ^w containing nodes x and y. By Lemma 13.8, X is a bridge of Type c with respect to Σ'. Since no node of A_4^w is adjacent to any node in $V(P) \cup V(Q)$, h_6' must be adjacent to x_j, and by Lemma 13.7 every node in A_6^w has a neighbor in X. In particular h_6 is adjacent to X, hence $j = n$. Therefore no node of $V(P) \cup V(Q) \setminus \{x_n\}$ is adjacent to any node in Θ_B^w. Let Y be a chordless path from h_1 to x in $V(P) \cup V(Q) \setminus \{x_n\}$. If h_3 is adjacent to Y, then Y induces a fan top with center h_3 contradicting the maximality of w. Else let X be a direct connection from h_3 to Y in the graph induced by $V(P) \cup V(Q) \setminus \{x_n\}$. If X has a neighbor of h_1 or x, then there is a fan top with center h_1 or x contradicting the maximality of w. Otherwise some subset of $V(X) \cup V(Y)$ induces a triad top contradicting the maximality of w. □

Proof of Theorem 13.3: If the edge set $E(A)$ does not disconnect Θ_T^w from Θ_B^w then the subgraph G obtained by removing the nodes $V(\Theta^w)$ contains a connected component S having at least one node adjacent to a node in Θ_T^w and at least one node adjacent to a node in Θ_B^w. Let $N(S)$ be the set of nodes of $V(\Theta^w)$ adjacent to at least one node in S.

Claim: $N(S) \cap (A_1^w \cup A_3^w \cup A_5^w) \neq \phi$ and $N(S) \cap (A_2^w \cup A_4^w \cup A_6^w) \neq \phi$.

Proof of Claim: Suppose that $N(S) \cap (A_1^w \cup A_3^w \cup A_5^w) = \phi$. Then S contains a path $P = x_1, \ldots, x_n$ which is a direct connection from $\Theta_T^w \setminus (A_1^w \cup A_3^w)$ to Θ_B^w avoiding $A_1^w \cup A_3^w$ in $G \setminus E(A)$, such that x_1 is adjacent to a node of $\Theta_T^w \setminus (A_1^w \cup A_3^w \cup A_5^w)$. This contradicts Lemma 13.4. Hence $N(S) \cap (A_1^w \cup A_3^w \cup A_5^w) \neq \phi$. Similarily $N(S) \cap (A_2^w \cup A_4^w \cup A_6^w) \neq \phi$. This completes the proof of the Claim.

So we only need to consider the following two cases.

Case 1: For every $u \in A_1^w \cup A_3^w \cup A_5^w$ and $v \in A_2^w \cup A_4^w \cup A_6^w$ such that $u, v \in N(S)$, uv is an edge.

Then for some $i \in \{1, \ldots, 6\}$, $V(A) \cap N(S) \subseteq A_{i-1}^w \cup A_i^w \cup A_{i+1}^w$. W.l.o.g. assume $i = 2$. By Theorem 12.11 the node set $K = A_1^w \cup A_2^w \cup A_3^w$ induces a biclique. Now we show that K is a biclique articulation seperating S from Θ^w. Suppose not. Then S contains a path $P = x_1, \ldots, x_n$ such that either P is a direct connection between $\Theta_T^w \setminus (A_1^w \cup A_3^w)$ and Θ_B^w avoiding $A_1^w \cup A_3^w$ in $G \setminus E(A)$ and x_1 is adjacent to a node of $\Theta_T^w \setminus (A_1^w \cup A_3^w \cup A_5^w)$, or P is a direct connection between $\Theta_B^w \setminus (A_2^w \cup A_6^w)$ and Θ_T^w avoiding $A_2^w \cup A_6^w$ in $G \setminus E(A)$ and x_n is adjacent to a node of $\Theta_B^w \setminus (A_2^w \cup A_4^w \cup A_6^w)$. In either case P contradicts Lemma 13.4.

Case 2: There are nodes $u \in A_1^w \cup A_3^w \cup A_5^w$ and $v \in A_2^w \cup A_4^w \cup A_6^w$ such that $u, v \in N(S)$ and uv is not an edge.

W.lo.g. assume that $N(S) \cap A_3^w \neq \phi$ and $N(S) \cap A_6^w \neq \phi$. Then there exists a path $P = x_1, \ldots, x_n$ which is a direct connection from $\Theta_T^w \setminus (A_1^w \cup A_5^w)$ to $\Theta_B^w \setminus (A_2^w \cup A_4^w)$ avoiding $A_1^w \cup A_5^w \cup A_2^w \cup A_4^w$ in $G \setminus E(A)$. Let $p \in \Theta_T^w \setminus (A_1^w \cup A_5^w)$ be adjacent to x_1 and $q \in \Theta_B^w \setminus (A_2^w \cup A_4^w)$ be adjacent to x_n. By Corollary 12.18, let Σ be a connected 6-hole of Θ^w containing p and q. By Lemma 13.8 P is a bridge of Type c with respect to Σ. W.l.o.g. assume P is adjacent to nodes h_1, h_2, h_3 and h_6. Now by Lemma 13.9 $A_1^w \cup A_2^w \cup A_3^w \cup N(h_2)$ is an extended star cutset seperating x_1 from Θ^w. □

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